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Jerry Michael Schirmer

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**Boundary conditions for Black Holes using the
Ashtekar Isolated and Dynamical Horizons
formalism**

Committee:

Richard Matzner, Supervisor

Duane Dicus

Sonia Paban

Dan Knopf

Philip Morrison

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by

Jerry Michael Schirmer, B.A.

Dissertation

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Dedicated to the generosity that I have received over the course of the past year, while my life has been turned upside down:

From Casey Clough, who kept the household running while I was moving back and forth and never in one place, and for always being ready with some delicious noodles or a pot of tea. Even after the most rough of work days.

From Jennifer Kreft-Pierce and Austin Pearce, who opened their home to me when I was under duress, and whose help I am incapable of understating.

From Allyson Whipple, during the afternoons at Mozarts and Bouldin Creek working on this, and who always had an ear to lend to me.

And from my parents, Anne and Michael Schirmer, who have always been there for me, no matter what.

Boundary conditions for Black Holes using the Ashtekar Isolated and Dynamical Horizons formalism

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Jerry Michael Schirmer, Ph.D.

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Supervisor: Richard Matzner

Isolated and Dynamical horizons are used to generate boundary conditions upon the lapse and shift vectors. Numerous results involving the Hamiltonian of General relativity are derived, including a self-contained derivation of the Hamiltonian equations of general relativity using both a direct 'brute force' method of directly computing Lie derivatives, as well as the standard Hamiltonian approach. Conclusions are compared to numerous examples, including the Kerr, Schwarzschild-De Sitter, McVittie, and Vaidya spacetimes.

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Chapter 1

Overview

1.1 Conventions

First, a basic note on convention. For what follows, we will be taking the signature $(-, +, +, +)$ for the metric tensor and the convention $(\nabla_a \nabla_b - \nabla_b \nabla_a)v_c = R_{abc}{}^d v_d$ for the Riemann tensor. We will also take $\epsilon_{0123} = +1$. If a mathematical quantity (i.e., R_{ab}, ∇_a , etc.) is written unmodified, it will be considered to be a 4-dimensional object. If the same object has a tilde placed over it (i.e., $\tilde{\nabla}_i$), then that object will be considered to be an object intrinsic to a null hypersurface embedded in a Minkowskian 4-space. If that same object has an overbar (\bar{R}_{ij}), then that object will be considered to be an object intrinsic to a 3-dimensional spacelike submanifold. If the object in question is denoted with an overhat (\hat{R}_{AB}), then that object will be considered to be intrinsic to a two-dimensional submanifold to the 4-dimensional space. We will also take the convention that indices that run over 4-dimensional labels will be labeled

with lower case Latin letters from the beginning of the alphabet (a,b,c,...). 3-dimensional indices labeled in the intrinsic coordinates of the 3-dimensional space will be labeled with lower case Latin letters running from the middle of the alphabet (i,j,k,...). Finally, 2-dimensional indices running over the coordinates intrinsic to a 2-dimensional space will be denoted by capital Latin letters running from the beginning of the alphabet (A,B,C,...). Note that, with the use of the inclusion operators to be defined below, it **is** possible to have (for instance) a 2-dimensional quantity whose indices are 4-dimensional.

1.2 Scope of Dissertation

Over the past three and a half decades, there has been a broad array of work on formalizing and redefining the notion of the black hole horizons, starting with the work of Regge and Tietelboim [1]. The primary thrust of this work has been to take the heavy mathematical machinery of immersions and manifolds and to use them to obtain a broad swath of powerful results in Relativity. However, as this has progressed, there has only recently been a focus upon how to compute and analyze these quantities. And while there are numerous attempts to use these quantities to set boundary conditions for numerical simulations [2] [3] [4] [5] [6] and for quantum gravitational computations [7], there has not been a clear method to use 3+1 conditions in order to track the motion of a dynamical horizon across a 3+1 slice. In section 8.1.1, this is precisely what is done. It is also explicitly shown, using the Vaiyda solution, how this term can be used to track the area growth of a black hole apparent

horizon without assuming any prior knowledge about the dynamics of the black hole.

This work, therefore, will have two goals. The first one will be to re-derive existing results in a way that makes it transparent how one can do concrete calculations using the listed techniques. Theorems from differential topology and geometry will be used with a focus towards indicating how one can choose a particular coordinate system and calculate quantities. In particular, with null geometry, this question is treated with great care, as a null surface will have a tangent space and a cotangent space that are not metrically related to each other. Thus, this work has many explicit examples and concrete calculations, both in the body of the paper and in the appendices. The explicit null decomposition of Minkowski spacetime and of the Kerr spacetime done in Appendix [A](#) is, to the author's knowledge, not currently published in the literature. The formalism used in doing this decomposition actually improves existing literature, as it resolves an ambiguity in the definition of the degenerate metric tensor and the metric compatible null connection noted by Ashtekar et al. [\[8\]](#)

Additional original work provided here includes the analysis of the McVittie spacetime^{[1](#)}. Discussion of the effects of the expansion of the universe on geodesics is described, as well as actual numerical computation of the geodesics. There is a discussion regarding the singularity of this spacetime

¹The McVittie spacetime fuses properties of the Schwarzschild and Robertson-Walker spacetimes, and thus can be considered to describe a black hole sitting in a Robertson-Walker background.

at the surface $r = 2M$, where curvature invariants $C_{abcd}C^{abcd}$, $R_{ab}R^{ab}$ and R are computed in order to establish the still-disputed claim that the McVittie “horizon” represents a spacetime singularity, and not a coordinate singularity á la Schwarzschild. Then the spacetime is used to generate results related to the Hamiltonian dynamics of Relativity, in particular, as an example of how one may make sense of an ADM mass even in a non-asymptotically flat spacetime.

This work also treats the elegant classical counterterm treatment of the ADM quantities, as originally computed by Hawking et al. rather than the more standard treatment which rejects boundary terms in a variational principle, and then inserts them at the last step in order to ensure consistency of the Hamiltonian equations of motion with the Hamiltonian [1]. While the treatment of Hamiltonian Relativity using counterterms is not original, the discussion to follow does provide a new spin upon the discussion, focused on the interaction between boundary degrees of freedom and bulk degrees of freedom.

Finally, this work derives boundary conditions upon the lapse function and a component of the shift vector, and does so both in the intrinsic language of the Isolated and Dynamical horizons of Ashtekar and in the 3+1 Hamiltonian language that is best suited for numerical simulations of Einstein’s equation. And this is all done with an aim of developing a clear technique to show how the relevant objects can actually be evaluated and computed in a 3+1 context.

Chapter 2

Basic Definitions and Decomposition Process

2.1 Choice of a 3+1 slice and normal vector

In the literature, there has been a dearth of clarity on the topic of projecting tensors from higher dimensional spaces onto lower dimensional spaces. In this section, a consistent procedure for doing this will be described, and then a direct application to the derivation of the (3+1) ADM split will be given. The key point to remember is that the projection operation has different behavior depending on whether one is using the projection map or the inclusion map and whether or not these maps are acting on vectors or one-forms.

Now, consider a decomposition of a general manifold \mathbb{M} with metric tensor g_{ab} . Let \mathbb{M} have the local topology $\mathbb{R} \times \mathbf{m}$, where \mathbf{m} is a submanifold to \mathbb{M} . Now, let τ be some function defined on \mathbb{M} which takes a constant value on

each fibre \mathbf{m} of \mathbb{R} . Then, $\nabla_a \tau$ will be a normal vector to the fiber. For now, assume that $\nabla_a \tau \nabla^a \tau \neq 0$, (a case that will be dealt with on page 72), and define ξ as the sign of $\nabla_a \tau \nabla^a \tau$. After making the definition $\alpha \equiv \frac{1}{\sqrt{\xi \nabla^a \tau \nabla_a \tau}}$, it is simple enough to define the unit normal to \mathbf{m} by the formula

$$n_a \equiv \alpha \nabla_a \tau. \quad (2.1.1)$$

Typically, the next step would be to go and define the normal projection operator $\gamma_{ab} \equiv g_{ab} - \xi n_a n_b$, which we will soon do, but some care should be taken with this definition, as there are two different senses in which we can take the phrase “3-metric”. First, we wish to choose a special coordinate system for our manifold. In particular, we wish to choose τ as one of the coordinates in the neighborhood of one of the fibers. Clearly, this should be a valid coordinate choice, as τ is constant on each fiber, so advancing in τ merely advances from one fibre to the next, just as stepping forward in t in standard Minkowski spacetime merely advances one from one spacelike copy of \mathbb{R}^3 to the next. So, after replacing one of the original manifold coordinates with τ we have a coordinate system of the form (τ, x^i) , where the index i ranges over the remaining coordinates in the spacetime.¹ Therefore, it is now clear that, by assumption, we have $\nabla_a \tau = (\xi, 0, 0, \dots, 0)$ ², and $\alpha = \frac{1}{\sqrt{\xi g^{\tau\tau}}}$. Therefore, this gives a relatively simple coordinate expression for the raised form of n^a :

¹While the language of this example is geared for the most common case, a timelike $\nabla_a \tau$ in a 4-dimensional Minkowskian spacetime, and the argument will advance as such, note that no such assumption is being made here. The above methodology will work for any non-null $\nabla_a \tau$.

²The factor of ξ is inserted so as to make $g^{ab} \nabla_a \tau$ future pointing

$$n^a = \frac{\xi}{\sqrt{\xi g^{\tau\tau}}} g^{\tau a}. \quad (2.1.2)$$

We therefore have an expression for the raised operator γ^{ab} in terms of the original metric components:

$$\gamma^{ab} = g^{ab} - \xi n^a n^b = g^{ab} - \xi \frac{1}{\xi g^{\tau\tau}} \xi g^{\tau a} \xi g^{\tau b} = g^{ab} - \frac{1}{g^{\tau\tau}} g^{\tau a} g^{\tau b} \dots \quad (2.1.3)$$

Where we use the fact that $\xi = \pm 1$ to set $\xi^2 = 1$. It can now clearly be seen that all of the components of γ^{ab} where either index takes on the value τ vanishes. It is therefore natural to consider γ^{ab} to be an operator that takes four-component one-forms on \mathbb{M} and projects them onto three-component vectors on \mathbf{m} . Therefore, we can consider the operator γ^{ab} in two senses: First, we can think of it as a projection operator living in a 4-dimensional space whose output is vectors with three nonzero entries. Secondly, we can think of it as an intrinsic 3-dimensional object that serves to define the inner product over the covector space to \mathbf{m} . Now, we will show that the natural inverse of the three by three matrix γ^{ij} is in fact obtained simply by restricting the 4-metric to the appropriate 3-dimensional indices, g_{ij} :

$$\gamma^{ij} g_{jk} = (g^{ij} - \frac{1}{g^{\tau\tau}} g^{\tau i} g^{\tau j}) g_{jk}, \quad (2.1.4)$$

$$= g^{ia} g_{ak} - g^{i\tau} g_{\tau k} - \frac{1}{g^{\tau\tau}} g^{\tau i} (g^{\tau a} g_{ak} - g^{\tau\tau} g_{\tau k}), \quad (2.1.5)$$

$$= \delta^i_k - g^{i\tau} g_{\tau k} - \frac{g^{\tau i}}{g^{\tau\tau}} \delta^\tau_k + g^{\tau i} g_{\tau k}, \quad (2.1.6)$$

$$= \delta^i_k. \quad (2.1.7)$$

Here, we change the summation over 3-dimensional indices to a summation over 4-dimensional indices by the identity $v^i v_i = v^a v_a - v^\tau v_\tau$, and, in the transition to the last line, we realize that k ranges over only spatial indices, so therefore δ^τ_k is identically zero, and we use the symmetry of the metric tensor to cancel the first and third terms. Therefore, we can see that the projection operator on covariant indices merely requires that we drop the τ component of the one-form, and this is where care should be taken, as if we use g^{ab} to raise and lower the indices of γ^{ab} , we could naïvely believe that the proper three projection of a one-form ω_a might be $\gamma^a_b \omega_a$ ³. A simple computation, however, will show that this form does, in fact, have a τ component, and therefore, is not properly a 3-dimensional one-form, in the sense that it is three quantities along the three coordinate basis one-forms on the cotangent space to \mathbf{m} . Instead, as the above example involving the metric indicates, the proper thing to do is, instead, simply drop the $d\tau$ component of the covector ω_a .

Finally, we might want to project a 4-dimensional vector onto our 3-dimensional space. This action, however, has a more complicated form than simply dropping the τ component. In order to accomplish this, we first lower the index of the vector, follow the above projection procedure, and then raise the vector with γ^{ij} :

³And, as we will see on page 12, in a certain sense it is

$$v^a \rightarrow g_{ia} v^a, \quad (2.1.8)$$

$$g_{ia} v^a = g_{i\tau} v^\tau + g_{ij} v^j, \quad (2.1.9)$$

$$\bar{v}^k = \gamma^{ki} g_{i\tau} v^\tau + \gamma^{ki} g_{ij} v^j, \quad (2.1.10)$$

$$= (g^{ki} + n^k n^i) g_{it} v^t + \gamma_j^k v^j, \quad (2.1.11)$$

$$= (\delta_\tau^k + n^k n_\tau) v^\tau + \gamma_j^k v^j, \quad (2.1.12)$$

$$= \gamma_a^k v^a. \quad (2.1.13)$$

So, while the operator γ_a^b does not take four component one forms on \mathbb{M} and spit out three component one forms on \mathbf{m} , it *does* do this to vectors.

On the other hand, it should be clear that we might have a problem going in the other direction, from 3-dimensional space to 4-dimensional space. In particular, if we were to map from the 3-space back into the 4-space, we would want our target vector/one-form to be normal to the appropriate version of n_a , since we want to consider n_a to be the normal to the 3-space. But, if we were to merely map a general 3-dimensional one form in the trivial manner:

$$s_i = (x_1, x_2, \dots, x_n) \rightarrow s_a = (0, x_1, x_2, \dots, x_n). \quad (2.1.14)$$

We can see right away that we will get a result that is inconsistent with our desire to be normal to n^a , since $n^a = \frac{-\xi}{\sqrt{\xi g^{tt}}} g^{ta}$ and, in general, the $g^{ti} \neq 0$. So, what are we to do? The answer is that, for one-forms, we use the operator γ_a^b not as a projection operator from \mathbb{M} onto \mathbf{m} , but rather as an *inclusion*

operator of \mathbf{m} into \mathbb{M} . If we use the above mapping, and then follow that with having the one-form be acted upon by γ^a_b , we get a vector that is normal to n^a , since

$$n^a \gamma_a^b = n^a (\delta_a^b + n_a n^b) = n^b - n^b = 0. \quad (2.1.15)$$

Furthermore, we have

$$\gamma^{ab} \gamma_{bc} = (g^{ab} + n^a n^b) (\gamma_{bc} + n_b n_c) = \delta^a_c + n^a n_c + n^a n_c - n^a n_c = \delta^a_c + n^a n_c = \gamma^a_c. \quad (2.1.16)$$

Now, we will show that use of γ^{ab} enables us to find a vector in \mathbb{M} that is, in a sense, equivalent to a one form living in the cotangent space of \mathbf{m} , thereby avoiding this problem. Then, once we know this duality, we can then use the appropriate metric tensor to raise and lower indices as we will. First, consider a one-form s_i living in \mathbf{m} . Define \bar{s}_a according to the trivial inclusion map seen in (2.1.14). Since we know that the components of g_{ij} are equivalent to the components of γ_{ij} , and that the matrix γ^{ij} is the inverse of the matrix γ_{ij} , while, at the same time, we know that γ_{ab} satisfies the identity

$$\begin{aligned} \gamma^{ab} \gamma_{bc} &= (g^{ab} - \xi n^a n^b) (g_{bc} - \xi n_b n_c), \\ &= \delta^a_c - \xi n^a n_c - \xi n^a n_c + \xi^3 n^a n_c, \\ &= \delta^a_c - \xi n^a n_c, \end{aligned}$$

$$=\gamma^a_c. \quad (2.1.17)$$

Then, we can show that if we define $s^a \equiv \gamma^{ab}\bar{s}_b$, we can show that the norm of \bar{s}_i under γ^{ij} is equivalent to the norm of s^a under g_{ab} :

$$\bar{s}_i\bar{s}^i = \gamma^{ij}\bar{s}_i\bar{s}_j, \quad (2.1.18)$$

$$= \gamma^{ab}\bar{s}_a\bar{s}_b, \quad (2.1.19)$$

$$= \gamma_{cd}\gamma^{ca}\gamma^{db}\bar{s}_a\bar{s}_b, \quad (2.1.20)$$

$$= (g_{cd} - \xi n_c n_d)\gamma^{ca}\bar{s}_a\gamma^{db}\bar{s}_b, \quad (2.1.21)$$

$$= g_{cd}s^c s^d, \quad (2.1.22)$$

$$= s^a s_a. \quad (2.1.23)$$

So, then, we can lower the left hand index of γ^{ab} using g_{ab} , and then, we can use the result, γ_a^b , as an inclusion map whose target has the same norm in \mathbb{M} as the source had in \mathbf{m} . Also, note that since n_a has only a τ component, we can simply drop the τ component of s_a to recover \bar{s}_i . Henceforward, this paper will drop the bar notation on one forms, and let the type of index denote whether we are discussing the 3-dimensional one-form or its inclusion into the four-manifold.

Meanwhile, using the trivial inclusion map for vectors creates no problem, since $\nabla_a \tau$ and n_a have no components along any other direction than the τ coordinate. Therefore, if we take:

$$s^i = (\bar{x}^1, \bar{x}^2 \dots \bar{x}^n) \rightarrow s^a = (0, \bar{x}^1, \bar{x}^2 \dots \bar{x}^n). \quad (2.1.24)$$

We are guaranteed to not run into any trouble, since the right hand of the above equation is normal to all of the one-forms with no components that project along the space, and it is guaranteed to have the same norm, since $g_{ij} = \gamma_{ij}$. So, we have the following rules for the projection and inclusion maps:

1. If we are using the projection map:
 - (a) One forms project trivially: just drop the appropriate component, and the projection is done
 - (b) Vectors, however, do NOT project trivially, and need to be projected using the γ^a_b operator, which is equivalent to lowering with the full metric, operating with the trivial projection map, and then raising with the three-metric
2. If we are acting with the inclusion map (we are taking 3-component objects intrinsic to a 3-surface, and seeing how they look in the full spacetime)
 - (a) in this case, it is the vectors who have the trivial behaviour. Just add a zero in the appropriate component, and then you have a vector in the full tangent space

- (b) One forms, however, need to have the inclusion matrix $\gamma_a{}^b$ operate on them. This is ultimately equivalent to raising them using the induced metric γ^{ij} , operating with the trivial inclusion map for vectors and then lowering with the enveloping four-dimensional metric.

The above procedure is justified by a look at any differential topology textbook (see, e.g., [9]), which will define the action of the pullback operator (that maps from a manifold with higher dimension to one with lower dimension) in terms of the differential of the map from the higher dimensional space to the lower dimensional space. It will therefore have a trivial action on forms. Meanwhile, the book will define the inclusion operator (which acts in the opposite direction) in terms of the vectors in the target space that are being acted on by the above forms, and therefore, the inclusion operator will act trivially on vectors, but not forms. The topology book will typically stop short of saying what to do with the non-trivial actions, as a metric tensor is typically not assumed. In our case, however, we can use the metric and induced metric to move the tangent space into the cotangent space, and then operate with the inclusion map and the projection map, and then move back between the tangent space and cotangent space to get the correct operator.

So, in summary, there are two different senses in which we can talk about 3-dimensional tensor indices. The first is to consider the tensors as arising from three component indices intrinsic to the 3-dimensional submanifold. The other is to consider the tensors as being the targets of the inclusion map

described above. These objects have four indices, and live in the 4-dimensional spacetime, but are “3-dimensional” in the sense that they are normal to the vector n^a and also to the one-form n_a . Above, we have shown that the two methods are equivalent, and have shown the manner in which we can map from one version of a ‘three-vector’ to another. From here on out, we will work primarily with the version of a three vector that lives in the 4-dimensional space, and then project down when necessary.

Finally, one might object to our above procedure, since we took the ten metric tensor components, and projected them down to get the lapse α , as defined in 2.1 and the three-metric γ_{ab} as defined in (2.1.3), which has six independent components. What of the other three 4-metric components? These three components are encoded in the shift vector, defined according to:

$$\beta_i \equiv g_{\tau i} . \tag{2.1.25}$$

It should be clear that this vector does, in fact, have three components that correspond to the metric tensor components that are dropped when the projection is done. And with that, we have encoded all of the ten independent metric tensor components in terms of equivalent 3-dimensional quantities. At times, it is convenient to speak of the time evolution vector, which is the vector that generates the evolution in the three-geometry as one advances along surfaces with increasing values of τ :

$$t^a \equiv \alpha n^a + \gamma^{ab} \beta_b . \tag{2.1.26}$$

In fact, it is easy to show that, in our coordinate system, this vector has only a τ component:

$$t^a = \alpha n^a + \gamma^{ab} \beta_b, \quad (2.1.27)$$

$$= \left(\frac{1}{\sqrt{\xi g^{\tau\tau}}} \right) \frac{\xi g^{\tau a}}{\sqrt{\xi g^{\tau\tau}}} + \gamma^{ab} g_{b\tau}, \quad (2.1.28)$$

$$= \frac{\xi g^{\tau a}}{\xi g^{\tau\tau}} + g^{ab} g_{b\tau} - \xi n^a n^b g_{b\tau}, \quad (2.1.29)$$

$$= \frac{g^{\tau a}}{g^{\tau\tau}} + \delta^a_\tau - \xi n^a n_\tau, \quad (2.1.30)$$

$$= \frac{g^{\tau a}}{g^{\tau\tau}} + \delta^a_\tau - \xi \left(\frac{\xi g^{\tau a}}{\sqrt{\xi g^{\tau\tau}}} \right) \frac{\xi}{\sqrt{\xi g^{\tau\tau}}}, \quad (2.1.31)$$

$$= \delta^a_\tau + \frac{g^{\tau a}}{g^{\tau\tau}} - \frac{\xi^3 g^{\tau a}}{\xi g^{\tau\tau}}, \quad (2.1.32)$$

$$= \delta^a_\tau. \quad (2.1.33)$$

Now, let us consider the appropriate way to define a 3-dimensional derivative operator. We remember from basic differential geometry that a derivative operator is unique if it (i) has domain and range within the manifold, (ii) maps the metric tensor to zero, and is (iii) torsion free. Therefore, consider the following definition, for any vector v_a such that $v_a n^a = 0$:

$$\bar{\nabla}_a v_b = \gamma_a^c \gamma_b^d \nabla_c v_d. \quad (2.1.34)$$

Condition (iii) is automatically satisfied if the original connection is torsion free, since you can't introduce antisymmetry in the Christoffel symbols by

multiplying them by matrices. Condition (i) is satisfied due to a combination of the restriction on v_a and equation (2.1.15) above. Appealing to the metric compatibility of the original connection, we can explicitly show that condition (ii) is satisfied:

$$\bar{\nabla}_a \gamma_{bc} = \gamma^m{}_a \gamma^n{}_b \gamma^k{}_c \nabla_m (g_{nk} + n_n n_k), \quad (2.1.35)$$

$$= 0 + \gamma^m{}_a \gamma^n{}_b \gamma^k{}_c (n_n \nabla_m n_k + n_k \nabla_m n_n), \quad (2.1.36)$$

$$= 0. \quad (2.1.37)$$

Since $\gamma^a{}_b$ annihilates n_a . Therefore, $\bar{\nabla}_a$ is the unique derivative operator on \mathbf{m} . Furthermore, we can note that this is, in fact, just equivalent to the standard definition one would make of a covariant derivative, if we first note that the operator $\gamma^a{}_b$ merely serves to project out any derivatives along the t direction. We can therefore take $\gamma_i^a \gamma_j^b \partial_a v_b = \partial_i v_j$ after having dropped the t indices according to the procedure outlined above. Now, we can check the equivalence of the definition (2.1.34) with the standard definition of the covariant derivative:

$$\begin{aligned} \gamma_i^a \gamma_j^b \nabla_a v_b &= \gamma_i^a \gamma_j^b (\partial_a v_b - \Gamma_{ab}^c v_c), \\ &= \partial_i v_j - \frac{1}{2} \gamma_i^a \gamma_j^b v_k \gamma^k{}_c g^{cm} (g_{am,b} + g_{bm,a} - g_{ab,m}), \\ &= \partial_i v_j - \frac{1}{2} \gamma_i^a \gamma_j^b v_k \gamma^{km} [\partial_a (\gamma_{bm} - n_b n_m) + \partial_b (\gamma_{am} - n_a n_m)] \end{aligned}$$

$$\begin{aligned}
& -\partial_k(\gamma_{ab} - n_a n_b)] , \\
& = \partial_i v_j - \frac{1}{2} \gamma_i^a \gamma_j^b v_k \gamma^{km} [\gamma_{am,b} + \gamma_{bm,a} - \gamma_{ab,m}] , \\
& = \partial_i v_j - \bar{\Gamma}_{ij}^k v_k
\end{aligned}$$

Where, in the transition from the third to fourth lines, we saw, that when the partial derivative operator acted on the $n_m n_n$ terms and we expanded with the product rule, we saw that it was impossible to not contract one of the n_m on a corresponding γ^m_n , yielding zero. Similarly, when going to the last line, we realized that, if all of the quantities involved are 3-dimensional, then the γ_a^b operator is merely the identity operator on the 3-dimensional space, and we defined the $\bar{\Gamma}_{ab}^c$ operator to be the obvious Christoffel symbol of the 3-dimensional space.

Now, beyond just the basic metric and connection of the three manifold, we should consider the information about the embedding of \mathbf{m} in \mathbf{M} . In order to do this, we introduce the extrinsic curvature tensor according to the definition:

$$K_{ab} \equiv -\gamma_a^c \gamma_b^d \nabla_c n_d . \quad (2.1.38)$$

Where the second factor of γ_b^d can be dropped in favor of δ_b^d since those two tensors differ only by a factor of $n_b n^d$, which vanishes when contracted on the covariant derivative term since n_a has a fixed norm. This concludes our basic discussion of how to encode 4-geometry in terms of 3-geometry. In the next chapter, we will investigate how to translate the curvature and Einstein's

equation into three-dimensional language, using a more explicit (albeit less elegant) methodology than is typically used in the literature.

Chapter 3

ADM decomposition of Einstein's Equation

In this section, we will now proceed to take the Einstein Equation, and express it in terms of the 3-dimensional quantities described above. This can be (and in the literature, usually is) done by writing the Hilbert action $\frac{1}{16\pi G} \int R d^4x$ in terms of the 3-dimensional quantities, defining the conjugate momentum to γ_{ab} , taking a Legendre transformation of the Lagrangian in order to find the Hamiltonian, and then writing down Hamilton's equations of motion. In this chapter, we take a different approach, explicitly calculating Lie derivatives of appropriate quantities using a 3+1 split. While this approach is much less elegant than the Hamiltonian technique, it serves a few purposes. First, it gives desired results much more directly—there is no need to Legendre transform, define conjugate momenta, and the like. In the Hamiltonian approach, the equations you naturally get involve the canonical momentum

$\Pi^{ab} \equiv \frac{\sqrt{\gamma}}{16\pi G} (\gamma^{ab} K - K^{ab})$, and the equation for $\dot{\Pi}^{ab}$ must be used to derive an equation for \dot{K}_{ab} , which involves significant amounts of algebra involving factors of $\dot{\gamma}_{ab}$. In the brute force method, with little embellishment, one can directly calculate $\dot{\gamma}_{ab}$ and \dot{K}_{ab} . Second, while both techniques are available for splitting a Lorentzian 4-manifold of local topology $\mathbb{R} \times \mathbf{m}$, only the brute force technique used here is generalizable to the splitting a neighborhood of the boundary of a Riemannian 3-surface into its 2-dimensional boundary and a 1-dimensional normal space.

For completeness, and for the sake of investigating the boundary terms that the Hilbert action necessarily gives rise to, we will compute the ADM equations using the Hamiltonian picture in Chapter (4)

3.1 Constraint Equations

First, given an arbitrary four vector v_a , let us recall the definition of the Riemann Curvature tensor:

$$R_{abc}{}^d v_d \equiv 2\nabla_{[a} \nabla_{b]} v_c. \quad (3.1.1)$$

Since we defined the 3-dimensional connection above in equation (2.1.34), we also have a simple enough expression for the 3-dimensional curvature tensor, of an arbitrary 3-dimensional one-form ω_c (and using the metric compatibility of the 4-connection, combined with $n^a \gamma_a^b = 0$):

$$\bar{R}_{abc}{}^d \omega_d = 2\bar{\nabla}_{[a} \bar{\nabla}_{b]} \omega_c, \quad (3.1.2)$$

$$= 2\gamma^m{}_{[a} \gamma^n{}_{b]} \gamma^k{}_c \nabla_m (\gamma^r{}_n \gamma^s{}_k \nabla_r \omega_s), \quad (3.1.3)$$

$$= 2\gamma^m{}_{[a} \gamma^r{}_{b]} \gamma^s{}_c \nabla_m \nabla_r \omega_s + 2\gamma^m{}_{[a} \gamma^r{}_{b]} \gamma^k{}_c (\nabla_m \gamma^s{}_k) \nabla_r \omega_s \\ + 2\gamma^m{}_{[a} \gamma^n{}_{b]} \gamma^s{}_c (\nabla_m \gamma^n{}_r) \nabla_r \omega_s, \quad (3.1.4)$$

$$= \gamma_a{}^m \gamma_b{}^n \gamma_c{}^k R_{mnk}{}^d \omega_d - 2\xi \gamma^m{}_{[a} \gamma^r{}_{b]} \gamma^k{}_c n^s (\nabla_m n_k) \nabla_r \omega_s \\ - 2\xi \gamma^m{}_{[a} \gamma^n{}_{b]} \gamma^s{}_c n^r (\nabla_m n_n) \nabla_r \omega_s. \quad (3.1.5)$$

Before we proceed farther, we should derive some properties of the gradient of n_a . First, as asserted above, we know that $n^b \nabla_a n_b = 0$, since it is easy to show that it is equal to minus itself. Now, let us evaluate $n^a \nabla_a n_b$. Using Equation (2.1.1):

$$n^a \nabla_a n_b = n^a \nabla_a (\xi \alpha \nabla_b \tau), \quad (3.1.6)$$

$$= \xi n^a \nabla_b \tau \nabla_a \alpha + \xi \alpha n^a \nabla_a \nabla_b \tau, \quad (3.1.7)$$

$$= n^a \frac{n_b}{\alpha} \nabla_a \alpha + \xi \alpha n^a \nabla_b \nabla_a \tau, \quad (3.1.8)$$

$$= n^a n_b \nabla_a \ln(\alpha) + \alpha n^a \nabla_b \left(\frac{n_a}{\alpha} \right), \quad (3.1.9)$$

$$= n^a n_b \nabla_a \ln(\alpha) - \alpha n^a n_a \frac{\nabla_b \alpha}{\alpha^2} + n^a \nabla_b n_a, \quad (3.1.10)$$

$$= n^a n_b \nabla_a \ln(\alpha) - \xi \nabla_b \ln(\alpha), \quad (3.1.11)$$

$$= -\xi \nabla_b \ln(\alpha) (\delta_b{}^a - \xi n^a n_b), \quad (3.1.12)$$

$$= -\xi \gamma_{bc} \gamma^{ca} \nabla_a \ln(\alpha), \quad (3.1.13)$$

$$= -\xi \bar{\nabla}_b \ln(\alpha). \quad (3.1.14)$$

Finally, we can see that, since γ^{ab} annihilates all vectors parallel to n_a , we have

$$\gamma^{am} \gamma^{bn} \nabla_m n_n = \gamma^{am} \gamma^{bn} \nabla_m (\alpha \nabla_n \tau) = \alpha \gamma^{am} \gamma^{bn} \nabla_m \nabla_n \tau. \quad (3.1.15)$$

By the vanishing of the torsion tensor, K^{ab} is symmetric¹, and we have:

$$\nabla_a n_b = \xi n_a \bar{\nabla}_b \ln(\alpha) - K_{ab}. \quad (3.1.16)$$

Using these results, we can see that the second line of equation (3.1.5) vanishes, since it is a 3-dimensional antisymmetrization of (3.1.16). Using the fact that $n^a \omega_a = 0$, we use the product rule to replace (3.1.5) with the following. We then proceed by factoring out the factor of ω_s and multiplying by γ_{sd} :

$$\begin{aligned} \bar{R}_{abc}{}^d \omega_d &= \gamma_a^m \gamma_b^n \gamma_c^k R_{mnk}{}^d \omega_d + \xi \gamma_a^m \gamma_b^r \gamma_c^k \omega_s (\nabla_m n_k) \nabla_r n^s \\ &\quad - \xi \gamma_b^m \gamma_a^r \gamma_c^k \omega_s (\nabla_m n_k) \nabla_r n^s, \end{aligned} \quad (3.1.17)$$

¹Note that although K^{ab} is symmetric, $\nabla_a n_b$ clearly is not, as should be explicitly clear in equation (3.1.16). It is only the projection of $\nabla_a n_b$ onto the 3-surface that is symmetric.

$$=\gamma_a^m \gamma_b^n \gamma_c^k R_{mnk}{}^s \omega_s + \xi K_{ac} K_b^s \omega_s - \xi K_{bc} K_a^s \omega_s, \quad (3.1.18)$$

$$\bar{R}_{abcd} = \gamma_a^m \gamma_b^n \gamma_c^k \gamma_d^l R_{mnkl} - \xi K_{ad} K_{bc} + \xi K_{ac} K_{bd}, \quad (3.1.19)$$

$$\bar{R} = \gamma^{ac} \gamma^{bd} R_{abcd} - \xi K^{ab} K_{ab} + \xi K^2, \quad (3.1.20)$$

$$= R - 2\xi R_{ab} n^a n^b - \xi K^{ab} K_{ab} + \xi K^2. \quad (3.1.21)$$

Having come this far, now we have to assume that we are making a timelike slicing of our spacetime. For the rest of this section, let us therefore assume that $\xi = -1$, meaning that n^a is a timelike vector.

Now, after making the definition $\rho \equiv n^a n^b T_{ab}$ and recalling Einstein's equation, we can derive the following identity:

$$\begin{aligned} 8\pi T_{ab} n^a n^b &= R_{ab} n^a n^b - \frac{1}{2} R g_{ab} n^a n^b, \\ 8\pi \rho &= \frac{1}{2} (2R_{ab} n^a n^b - \xi R), \\ 16\pi \rho &= 2R_{ab} n^a n^b - \xi R, \\ 2R_{ab} n^a n^b &= 16\pi \rho + \xi R. \end{aligned} \quad (3.1.22)$$

Combining equations (3.1.21) and (3.1.22), we obtain the Hamiltonian constraint²:

²we set $\xi = -1$ in the last step to get the explicit form of the Hamiltonian constraint for a 3+1 split. Also, in the second to last line, both sides are multiplied through with a factor of ξ , and all factors of ξ^2 are set equal to 1.

$$\begin{aligned}
\bar{R} &= R - \xi (16\pi\rho + \xi R) + \xi (K^2 - K^{ab}K_{ab}) , \\
16\xi\pi\rho &= -\bar{R} + \xi (-K_{ab}K^{ab} + K^2) , \\
16\pi\rho &= -\xi\bar{R} - K^{ab}K_{ab} + K^2 , \\
16\pi\rho &= \bar{R} - K^{ab}K_{ab} + K^2 .
\end{aligned} \tag{3.1.23}$$

Now, using a slightly different technique, we can derive the Momentum Constraint. First, project the Einstein Equation onto $n^a\gamma^b_c$, and define $j_a \equiv -T_{bc}n^c\gamma^b_a$:

$$\begin{aligned}
8\pi T_{ab}n^a\gamma^b_c &= R_{ab}n^a\gamma^b_c - \frac{1}{2}g_{ab}n^a\gamma^b_c R , \\
-8\pi j_c &= n^a\gamma^b_c g^{de}R_{dbea} - \frac{1}{2}n_b\gamma^b_c R , \\
&= 2\gamma^b_c g^{de}\nabla_{[d}\nabla_{b]}n_e , \\
&= 2\gamma_c^b\nabla_{[d}\nabla_{b]}n^d = \gamma_c^b\nabla_d\nabla_b n^d - \gamma_c^b\nabla_b\nabla_d n^d , \\
&= \nabla_d(\gamma_c^b\nabla_b n^d) - (\nabla_d\gamma_c^b)\nabla_b n^d + \bar{\nabla}_c K , \\
&= -\nabla_d K_c^d + \xi n^b(\nabla_d n_c)\nabla_b n^d + \xi n_c(\nabla_d n^b)\nabla_b n^d + \bar{\nabla}_c K , \\
8\pi j_c &= \nabla_d K_c^d + \xi K_{cd}n^b\nabla_b n^d + \xi n_c K_d^b\nabla_b n^d - \bar{\nabla}_c K , \\
&= g^{bd}\nabla_d K_{bc} - \xi n^b n^d\nabla_b K_{cd} - \xi n_c n^d\nabla_b K^b_d - \bar{\nabla}_c K , \tag{3.1.24}
\end{aligned}$$

$$\begin{aligned}
&= (g^{bd} - \xi n^b n^d)(g_c^e - \xi n_c n^e)\nabla_d K_{be} - \bar{\nabla}_c K , \tag{3.1.25} \\
&= \bar{\nabla}_b K^b_c - \bar{\nabla}_c K ,
\end{aligned}$$

$$8\pi j^a = \bar{\nabla}_b (K^{ab} - \gamma^{ab} K). \quad (3.1.26)$$

Here, Equation (3.1.26) is the momentum constraint, and in moving from equation (3.1.24) to equation (3.1.25), we used the fact that $n^a n^b n^c n_d \nabla_a K_{bc} = n^a n^c n_d \nabla_a (n^b K_{bc}) - n^a n^c K_{bc} n_d \nabla_a n^b = 0 - 0^3$. We have now derived expressions for the Constraint equations for the standard ADM 3+1 split.

3.2 Evolution Equations

Now, we will derive expressions for the ADM evolution equations. Once again, rather than going with the more elegant Hamiltonian framework, we are instead going to explicitly evaluate the time derivative (in this case, the Lie derivative along the vector t^a defined in equation (2.1.26)) of our evolution variables γ_{ab} and K_{ab} . First, let us find the evolution equation for γ_{ab} :

$$\mathcal{L}_t \gamma_{ab} = \mathcal{L}_{\alpha n} \gamma_{ab} + \mathcal{L}_{\beta} \gamma_{ab}, \quad (3.2.1)$$

$$\begin{aligned} &= \alpha n^c \nabla_c \gamma_{ab} + \gamma_{cb} \nabla_a (\alpha n^c) + \gamma_{ac} \nabla_b (\alpha n^c) + \beta^c \bar{\nabla}_c \gamma_{ab} \\ &\quad + \gamma_{cb} \bar{\nabla}_a \beta^c + \gamma_{ac} \bar{\nabla}_b \beta^c, \end{aligned} \quad (3.2.2)$$

$$= \alpha [n_a n^c \nabla_c n_b + n_b n^c \nabla_c n_a + \nabla_a n_b + \nabla_b n_a] + 2 \bar{\nabla}_{(b} \beta_{a)}, \quad (3.2.3)$$

$$= \alpha [(n_a n^c + g_a^c) \nabla_c n_b + (n_b n^c + g_b^c) \nabla_c n_a] + 2 \bar{\nabla}_{(b} \beta_{a)}, \quad (3.2.4)$$

$$= -2\alpha K_{ab} + 2 \bar{\nabla}_{(a} \beta_{b)}. \quad (3.2.5)$$

³Note that the common factor here is included simply because it also appears in the derivation of Equation (3.1.26)

Since it is expressed entirely in terms of 3-dimensional quantities (namely, we've eliminated any reference to n^a), we can consider equation (3.2.5) to be the evolution equation for γ_{ab} . Note that no reference to the Einstein Equation was required in this derivation. We can consider this evolution equation, therefore, to be more of a definition of the relationship between t^a , K_{ab} and β_a than anything else.

Now, let us derive an expression for the evolution equation for K_{ab} :

$$\mathcal{L}_t K_{ab} = \mathcal{L}_{\alpha n} K_{ab} + \mathcal{L}_\beta K_{ab}, \quad (3.2.6)$$

$$= \mathcal{L}_{\alpha n} K_{ab} + \beta^c \bar{\nabla}_c K_{ab} + K_{cb} \bar{\nabla}_a \beta^c + K_{ac} \bar{\nabla}_b \beta^c, \quad (3.2.7)$$

$$(\mathcal{L}_t - \mathcal{L}_\beta) K_{ab} = \alpha n^c \nabla_c K_{ab} + K_{cb} \nabla_a (\alpha n^c) + K_{ac} \nabla_b (\alpha n^c), \quad (3.2.8)$$

$$= \alpha [-n^c \nabla_c (\gamma_a^d \gamma_b^e \nabla_d n_e) + K_{cb} \nabla_a n^c + K_{ac} \nabla_b n^c], \quad (3.2.9)$$

$$\begin{aligned} &= \alpha [-n^c \gamma_a^d \gamma_b^e \nabla_c \nabla_d n_e - n^c (\nabla_c \gamma_b^e) \gamma_a^d \nabla_d n_e \\ &\quad - n^c (\nabla_c \gamma_a^d) \nabla_d n_b - K_{cb} K_a^c - K_{cb} n_a \bar{\nabla}^c \ln(\alpha) \\ &\quad - K_{ac} K_b^c - K_{ac} n_b \bar{\nabla}^c \ln(\alpha)], \end{aligned} \quad (3.2.10)$$

$$\begin{aligned} &= \alpha [-n^c \gamma_a^d \gamma_b^e R_{cde}{}^f n_f - n^c \gamma_a^d \gamma_b^e \nabla_d \nabla_c n_e \\ &\quad + n^c n_b (\nabla_c n^e) K_{ae} - n^c n^d (\nabla_c n_a) \nabla_d n_b \\ &\quad - n^c n_a (\nabla_c n^d) \nabla_d n_b - 2K_{ac} K_b^c] - n_a K_b^c \bar{\nabla}_c \alpha \\ &\quad - n_b K_a^c \bar{\nabla}_c \alpha, \end{aligned} \quad (3.2.11)$$

$$\begin{aligned} &= \alpha [(\gamma^{cf} - g^{cf}) \gamma_a^d \gamma_b^e R_{dcef} - \gamma_a^d \gamma_b^e \nabla_d (n^c \nabla_c n_e) \\ &\quad + \gamma_a^d \gamma_b^e (\nabla_d n^c) \nabla_c n_e + n_b K_{ae} \bar{\nabla}^e \ln(\alpha) \end{aligned}$$

$$\begin{aligned}
& - (\bar{\nabla}_a \ln(\alpha)) \bar{\nabla}_b \ln(\alpha) + n_a K_{db} \bar{\nabla}^d \ln(\alpha) - 2K_{ac} K^c_b] \\
& - n_a K^c_b \bar{\nabla}_c \alpha - n_b K_a^c \bar{\nabla}_c \alpha, \tag{3.2.12}
\end{aligned}$$

$$\begin{aligned}
& = \alpha [\gamma_a^l \gamma^{nk} \gamma_b^m R_{lnmk} - \gamma_a^m \gamma_b^n R_{mn} \\
& - \gamma_a^d \gamma_b^e \nabla_d (\bar{\nabla}_e \ln(\alpha)) + K_a^c K_{cb} - 2K_{ac} K^c_b] \\
& - \frac{1}{\alpha} (\bar{\nabla}_a \alpha) \bar{\nabla}_b \alpha, \tag{3.2.13}
\end{aligned}$$

$$\begin{aligned}
& = \alpha [\gamma_a^l \gamma^{nk} \gamma_b^m R_{lnmk} - \gamma_a^m \gamma_b^n (8\pi T_{mn} + \frac{1}{2} R g_{ab}) \\
& - K_{ac} K^c_b] - \frac{1}{\alpha} (\bar{\nabla}_a \alpha) \bar{\nabla}_b \alpha - \alpha \bar{\nabla}_a (\frac{1}{\alpha} \bar{\nabla}_b \alpha), \tag{3.2.14}
\end{aligned}$$

$$\begin{aligned}
& = \alpha [\bar{R}_{ab} - K_{ac} K^c_b + K K_{ab} - 8\pi S_{ab} - \frac{1}{2} R \gamma_{ab} \\
& - K_{ac} K^c_b] - \bar{\nabla}_a \bar{\nabla}_b \alpha, \tag{3.2.15}
\end{aligned}$$

$$\begin{aligned}
& = \alpha [\bar{R}_{ab} - 2K_{ac} K^c_b + K K_{ab} - 8\pi S_{ab} + 4\pi \gamma_{ab} (S - \rho)] \\
& - \bar{\nabla}_a \bar{\nabla}_b \alpha, \tag{3.2.16}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_t K_{ab} & = \alpha [\bar{R}_{ab} - 2K_{ac} K^c_b + K K_{ab} - 8\pi S_{ab} + 4\pi \gamma_{ab} (S - \rho)] \\
& - \bar{\nabla}_a \bar{\nabla}_b \alpha + \beta^c \bar{\nabla}_c K_{ab} + K_{cb} \bar{\nabla}_a \beta^c + K_{ac} \bar{\nabla}_b \beta^c. \tag{3.2.17}
\end{aligned}$$

In going from lines (3.2.14) to the next line, the definition $T_{ab} \gamma^a_m \gamma^b_n \equiv S_{mn}$ was made, and the result from equation (3.1.19) was used. In going forward from (3.2.15), we contracted the Einstein Equation on g^{ab} in the following way:

$$g^{ab} R_{ab} - \frac{1}{2} g^{ab} g_{ab} R = 8\pi g^{ab} T_{ab}, \tag{3.2.18}$$

$$R - 2R = 8\pi(\gamma^{ab} - n^a n^b)T_{ab}, \quad (3.2.19)$$

$$-R = 8\pi(S - \rho). \quad (3.2.20)$$

Since, once again, equation (3.2.17) depends only on 3-dimensional quantities, we can consider it to be the proper evolution equation for K_{ab} , and we now have the complete set of ADM equations, namely equations (3.2.17), (3.2.5), (3.1.23) and (3.1.26), and this concludes our “brute force” derivation of the ADM equations.

Chapter 4

Hamiltonian General Relativity

Now, we above performed the brute force derivation of the ADM equations for a few reasons. First, there was the goal of seeing true completeness to our derivation—there was no direct reference to non-geometric quantities above—we were able to show that the ADM equations are a direct consequence of merely the Einstein Equation and the rules of Lorentizan geometry. Second, the above procedure generalizes quite nicely to a foliation of a spacelike two surface embedded in a spacelike three surface—the relevant situation to many initial value boundary problems in General Relativity.

This is not to say, however, that the more standard Hamiltonian approach to this derivation is without merit. It is only through this approach that we can investigate several important issues—in particular, the Hamiltonian formalism enables us to investigate the angular and linear momentum stored in a spacetime, in addition to that spacetime’s net energy. The brute force approach given above necessarily only deals directly with bulk terms—these are

things that are best viewed through an action principle. So, here, we will work through this action principle in general relativity.

4.1 From the Lagrangian to the Hamiltonian

We begin with the well-known (see, for instance, [10]) Hilbert action:

$$S = \frac{1}{16\pi G} \int \sqrt{-g} R d^4x \dots \quad (4.1.1)$$

Here, g_{ab} is the four-dimensional Minkowskian metric of the spacetime, g is the determinant of g_{ab} , and R is the trace of its Ricci curvature. We wish to take this Lagrangian density and form a Hamiltonian density with proper equations of motion. It is a known result that no second time derivatives of the g_{ta} appear in the action (4.1.1), and therefore, if we were to take all of the components g_{ab} to be our canonical variables, we would then define our canonical momenta to be $P^{ab} \equiv \frac{\delta S}{\delta \dot{g}_{ab}}$, but then we would be stuck, because since the Hessian

$$H^{abcd} \equiv \frac{\delta^2 S}{\delta \dot{g}_{ab} \delta \dot{g}_{cd}} = \frac{\delta P^{ab}}{\delta \dot{g}_{cd}} \dots \quad (4.1.2)$$

would have a vanishing determinant¹, meaning that it would be impossible to solve for all of the P^{ab} in terms of all of the \dot{g}_{ab} . In particular, in transforming

¹To see this, if we set a and c to t , then we know that the Hessian is zero, regardless of the value of b and d (there are no second time derivatives of the g_{ta}). This, in turn, then indicates that an entire row of the Hessian Matrix vanishes, which then implies that the determinant of the Hessian vanishes

the variables of our system from the g_{ab} and the \dot{g}_{ab} to the g_{ab} and the P^{ab} , there would be a noninvertible transformation operator, meaning that degrees of freedom would be lost in going from the g_{ab} to the P^{ab} . The Lagrangian and Hamiltonian formulations of this theory would therefore not be compatible.

In order to avoid this problem, we reduce the number of variables in our system to the g_{ij} , which, as shown in section 2, are exactly equal to the 3-components of the 3-dimensional metric γ_{ab} if we were to do a 3+1 decomposition using t as our slicing variable². Then, using Equations (2.1) and (2.1.25) to define α and β_i , a straightforward calculation relates g to the determinant γ of γ_{ab} :

$$g = -\alpha^2 \gamma. \quad (4.1.3)$$

Then, we use Equation (3.1.22) to replace R with \bar{R} , giving us:

$$S = \frac{1}{16\pi G} \int \alpha \sqrt{\gamma} (\bar{R} + \xi K_{ab} K^{ab} - \xi K^2 + 2\xi R_{ab} n^a n^b) d^4x. \quad (4.1.4)$$

Since \bar{R} manifestly has no time derivatives in it, and $\frac{\delta K_{ab}}{\delta \dot{g}_{cd}} = \frac{1}{2}(\delta_a^c \delta_b^d + \delta_a^d \delta_b^c)(-2\alpha)$, we can see that (4.1.4) brings us closer to the canonical form necessary to Legendre transform the Hilbert action: every term except for the last either has no time derivatives, or is expressible in terms of the \dot{g}_{ij} . So, what of that fourth term? Let us investigate it further:

²of course, if we want *another* slicing, we can make this choice simply by coordinate transforming t to some other, more general function τ with a timelike gradient

$$n^a n^b R_{ab} = n^a n^b g^{cd} R_{cadb}, \quad (4.1.5)$$

$$= n^a g^{cd} (\nabla_c \nabla_a n_d - \nabla_a \nabla_c n_d), \quad (4.1.6)$$

$$= n^a (\nabla_c \nabla_a n^c - \nabla_a \nabla_c n^c), \quad (4.1.7)$$

$$\begin{aligned} &= \nabla_c (n^a \nabla_a n^c) - (\nabla_c n^a) \nabla_a n^c - \nabla_a (n^a \nabla_c n^c) + (\nabla_a n^a) \nabla_c n^c, \\ &= \nabla_a (-\xi \bar{\nabla}^a \ln(\alpha)) + \nabla_a (n^a K) - K^{ab} K_{ab} + K^2. \end{aligned} \quad (4.1.8)$$

Before we progress further, we will quickly derive two identities. The first simplifies the term involving the lapse function:

$$\begin{aligned} \nabla_a \bar{\nabla}^a \ln(\alpha) &= \nabla_a (\gamma^{ab} \nabla_b \ln(\alpha)), \\ &= \nabla_a \left(\frac{1}{\alpha} \gamma^{ab} \nabla_b \alpha \right), \\ &= \frac{1}{\alpha} \gamma^{ab} \nabla_a \nabla_b \alpha + \frac{1}{\alpha} (\nabla_b \alpha) \nabla_a \gamma^{ab} - \frac{1}{\alpha^2} \gamma^{ab} (\nabla_a \alpha) (\nabla_b \alpha), \\ &= \frac{1}{\alpha} \left[\gamma^{ac} \gamma_c^b \nabla_a \nabla_b \alpha + \nabla_b \alpha \nabla_a \gamma^{ab} - \frac{1}{\alpha} (\bar{\nabla}^a \alpha) \bar{\nabla}_a \alpha \right], \\ &= \frac{1}{\alpha} \left[\gamma^{ac} \nabla_a (\gamma_c^b \nabla_b \alpha) - \gamma^{ac} (\nabla_a \gamma_c^b) \nabla_b \alpha + (\nabla_b \alpha) \nabla_a \gamma^{ab} \right. \\ &\quad \left. - \frac{1}{\alpha} (\bar{\nabla}_a \alpha) \bar{\nabla}^a \alpha \right], \\ &= \frac{1}{\alpha} \left[\bar{\nabla}^2 \alpha + \xi \gamma^{ac} \nabla_b \alpha (n_c \nabla_a n^b + n^b \nabla_a n_c) \right. \\ &\quad \left. - \xi (\nabla_b \alpha) (n^a \nabla_a n^b + n^b \nabla_a n^a) - \frac{1}{\alpha} (\bar{\nabla}^a \alpha) \bar{\nabla}_a \alpha \right], \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\alpha} \left[\bar{\nabla}^2 \alpha - 0 - \xi K n^b \nabla_b \alpha + \xi^2 \nabla_b \alpha \bar{\nabla}^b l n(\alpha) + \xi (\nabla_b \alpha) n^b K \right. \\
&\quad \left. - \frac{1}{\alpha} (\bar{\nabla}^a \alpha) \bar{\nabla}_a \alpha \right], \\
&= \frac{1}{\alpha} \bar{\nabla}^2 \alpha.
\end{aligned} \tag{4.1.9}$$

Which eliminates the 4-divergences of α and replaces them with 3-divergences.

The second identity simplifies the term that involves the total divergence of $n^a K$ ³ In the following derivation double overbarred terms like $\bar{\bar{\gamma}}$ will refer to the metric intrinsic to the boundary to the 4-dimensional manifold \mathbb{M} .⁴

$$\begin{aligned}
\int d^4x \sqrt{-g} \nabla_a (n^a K) &= \oint d^3x \sqrt{\bar{\bar{\gamma}}} s_a n^a K, \\
&= \oint d^3x \left(\frac{\sqrt{-g}}{B} \right) K g^{ab} n_a s_b, \\
&= \oint d^3x \alpha^2 \sqrt{\gamma} K g^{ab} (\nabla_a \tau) \nabla_b R, \\
&= \oint d^3x \alpha^2 \sqrt{\gamma} K \left(-\frac{1}{\alpha^2} \delta^b_\tau + \frac{1}{\alpha^2} \beta^b \right) \nabla_b R, \\
&= \int dt \oint d^2x \bar{B} \sqrt{q} K \beta^a \bar{\nabla}_a R,
\end{aligned}$$

³Here, we consider the boundary to \mathbb{M} to be divided into three parts—the initial and final timeslices of our evolution—namely, the endpoints of the function τ , and a part topologically equivalent to $\mathbb{R} \times \partial \mathbf{m}$. We will worry about the latter below, and take the former only as an integral over initial and final timeslices, which we shall neglect, as the initial and final states of our variation will be considered fixed.

⁴Also note that following the argument by York outlined in Appendix we must make a distinction between the normal to boundary in the 4-manifold and its projection onto a 3-manifold. The former will be labeled as $s_a = B \nabla_a R$, and the latter will be labeled as $\bar{s}_a = \bar{B} \bar{\nabla}_a R$. Note that while the two are defined by the same function R , they are quite distinct objects—the latter is normal to the unit timelike normal n^a , and the normalization constants are fixed, respectively, by the 4-metric g^{ab} and the 3-metric γ^{ab} .

$$\begin{aligned}
&= \int dt \oint d^2x \sqrt{q} K \bar{s}_a \beta^a , \\
&= \int dt \int d^3x \sqrt{\gamma} \bar{\nabla}_a (\beta^a K) , \\
&= \int d^4x \sqrt{-g} \frac{1}{\alpha} \bar{\nabla}_a (\beta^a K) . \tag{4.1.10}
\end{aligned}$$

So, therefore, this term is also equivalent to an appropriate integral over $\partial\mathbf{m}$ integrated over time, or equivalently, a 3-divergence integrated over the 4-manifold.

Now, we can combine equations (4.1.9) and (4.1.10) with their corresponding terms in (4.1.8) to get:

$$n^a n^b R_{ab} = -\xi \frac{1}{\alpha} \bar{\nabla}^2 \alpha + \frac{1}{\alpha} \bar{\nabla}_a (\beta^a K) - K^{ab} K_{ab} + K^2 . \tag{4.1.11}$$

Now, putting (4.1.11) back into the fourth term of (4.1.4) we get that the term involving $R_{ab} n^a n^b$ reduces to:

$$\begin{aligned}
&\frac{1}{16\pi G} \int \sqrt{|g|} d^4x (2R_{ab} n^a n^b) = , \\
&= \frac{2}{8\pi G} \int d^4x \alpha \sqrt{\gamma} [-\xi \bar{\nabla}^2 \alpha + \nabla_a (n^a K) - K^{ab} K_{ab} + K^2] \\
&+ \frac{1}{8\pi G} \oint d^3x s_a n^a K ,
\end{aligned}$$

$$= \frac{1}{8\pi G} \oint d^3x s_a n^a K + \frac{1}{16\pi G} \int d^4x \alpha \sqrt{\gamma} \left[2K^2 - 2K^{ab} K_{ab} - \frac{2\xi}{\alpha} \bar{\nabla}^2 \alpha \right] \dots \quad (4.1.12)$$

Finally, we can substitute (4.1.12) into the Hilbert action (4.1.4) to obtain

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{\gamma} [\alpha (\bar{R} - K^2 + K^{ab} K_{ab}) + 2\bar{\nabla}^2 \alpha] + \frac{1}{8\pi G} \oint d^3x s_a n^a K \dots \quad (4.1.13)$$

Where we have taken $\xi = -1$ again to reflect the timelike slicing of our spacetime in the bulk. However, the explicit inclusion of the factor of ξ in going to equation (4.1.12) shows us that there is no sign difference in the boundary term, no matter whether we are evaluating the boundary term on the initial timeslice or on the outer boundary of our spacetime. This is fortunate, as it avoids any contradictions on the intersection of the initial timeslice and the outer boundary.

Now, we can (due to our assumption of topology $\mathbb{R} \times \mathbf{m}$ for our spacetime) remove the integral over our time coordinate τ from (4.1.13), and infer the Lagrangian density⁵ :

⁵Note that the text has replaced the normal n^a with a term r^a . We do this in order to indicate that r_a is explicitly unit spacelike, as we mean for the Lagrangian density to be defined on a single timeslice, which has a timelike outer boundary, and therefore, this boundary has a spacelike normal. Also note that we have converted the boundary term in (4.1.13) with a divergence.

$$\mathcal{L} = \frac{1}{16\pi G} \alpha \sqrt{\gamma} (\bar{R} + K^{ab} K_{ab} - K^2) + \frac{1}{8\pi G} \sqrt{\gamma} \nabla_c (r^c K). \quad (4.1.14)$$

Note that (4.1.14) is now written explicitly in terms of γ_{ab} and K_{ab} , which is directly related to $\mathcal{L}_t \gamma_{ab}$, so we have successfully separated out the “good” canonical variables from the Lagrange multipliers. Therefore, we can now feel safe defining conjugate momenta and taking a Legendre transform. Recalling from the ADM equation for γ_{ab} , we have $\mathcal{L}_t \gamma_{ab} = \dot{\gamma}_{ab} = -2\alpha K_{ab}$, we therefore can now define the momentum conjugate to γ_{ab} :

$$\Pi^{ab} \equiv \frac{\delta \mathcal{L}}{\delta \dot{\gamma}_{ab}}, \quad (4.1.15)$$

$$= \frac{1}{8\pi G} \alpha \sqrt{\gamma} \left(\frac{1}{2\alpha} \right) (K \gamma^{ab} - K^{ab}) + \frac{1}{4\pi G} \frac{\delta}{\delta \dot{\gamma}_{ab}} \sqrt{\gamma} \nabla_c (r^c \gamma^{ab} K_{ab}), \quad (4.1.16)$$

$$= \frac{1}{16\pi G} \sqrt{\gamma} (K \gamma^{ab} - K^{ab}) \dots \quad (4.1.17)$$

Where we dropped the $\nabla_c (r^c K)$ term in equation (4.1.16) due to the fact that since $\bar{\nabla}_c \gamma^{ab} = 0$, we therefore have, after integrating by parts⁶:

⁶ Note that while it is true that $\frac{\delta K_{ab}}{\delta \dot{\gamma}_{mn}} = \delta_{ab}^{mn}$, this is not true of $\frac{\delta}{\delta \dot{\gamma}_{mn}} \nabla_c K_{ab}$. In the latter case, it is appropriate to interchange the functional derivative with the covariant derivative and integrate by parts. This originates from the fact that a formal definition of the variational derivative is based in taking terms proportional to differentials of functions. The entire functional derivative should be the multiplier of the differential of the function, which does not work if it is inside of a derivative.

$$\frac{\delta}{\delta \dot{\gamma}_{ab}} \int d^3x \sqrt{\gamma} \bar{\nabla}_c (r^c \gamma^{de} K_{de}) = \quad (4.1.18)$$

$$= \frac{\delta}{\delta \dot{\gamma}_{ab}} \int d^3x \sqrt{\gamma} \gamma^{de} \bar{\nabla}_c (r^c K_{de}), \quad (4.1.19)$$

$$= \frac{\delta}{\delta \dot{\gamma}_{ab}} \int d^3x \sqrt{\gamma} \gamma^{de} (K_{de} \bar{\nabla}_c (r^c) + r^c \bar{\nabla}_c K_{de}), \quad (4.1.20)$$

$$= \int d^3x \sqrt{\gamma} (\gamma^{de} (\bar{\nabla}_c r^c) \frac{\delta K_{de}}{\delta \dot{\gamma}_{ab}} - \bar{\nabla}_c (\gamma^{de} r^c) \frac{\delta K_{de}}{\delta \dot{\gamma}_{ab}}), \quad (4.1.21)$$

$$= 0. \quad (4.1.22)$$

So, now, we have our canonical variables γ_{ab} , their conjugate momenta Π^{ab} , and we are left with four Lagrange multipliers in α and the β_a . Inverting equation (4.1.17) to obtain expressions for K and K^{ab} in terms of Π^{ab} gives us:

$$K = \frac{8\pi G}{\sqrt{\gamma}} \Pi \quad K^{ab} = \frac{16\pi G}{\sqrt{\gamma}} \left(\frac{1}{2} \gamma^{ab} \Pi - \Pi^{ab} \right). \quad (4.1.23)$$

An obvious consequence of this is that:

$$\begin{aligned} K^2 &= \left(\frac{8\pi G}{\sqrt{\gamma}} \right)^2 \Pi^2, \\ K^{ab} K_{ab} &= \left(\frac{8\pi G}{\sqrt{\gamma}} \right)^2 (\gamma^{ab} \Pi - 2\Pi^{ab})(\gamma_{ab} \Pi - 2\Pi_{ab}), \\ &= \left(\frac{8\pi G}{\sqrt{\gamma}} \right)^2 (3\Pi^2 - 4\Pi^2 + 4\Pi^{ab} \Pi_{ab}), \end{aligned} \quad (4.1.24)$$

$$= \left(\frac{8\pi G}{\sqrt{\gamma}} \right)^2 (4\Pi^{ab}\Pi_{ab} - \Pi^2). \quad (4.1.25)$$

Giving us the useful result:

$$K^{ab}K_{ab} - K^2 = \left(\frac{16\pi G}{\sqrt{\gamma}} \right)^2 (\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2). \quad (4.1.26)$$

Knowing this⁷ we are ready to take the Legendre transform of our Lagrangian density \mathcal{L} in order to define a Hamiltonian density \mathcal{H} :

$$\mathcal{H} \equiv \Pi^{ab}\dot{\gamma}_{ab} - \mathcal{L}. \quad (4.1.27)$$

Remembering from equation (3.2.5) that $\mathcal{L}_t\gamma_{ab} = \dot{\gamma}_{ab} = -2\alpha K_{ab} + 2\bar{\nabla}_{(a}\beta_{b)}$, we can then explicitly perform this procedure:

$$\begin{aligned} \mathcal{H} &= \Pi^{ab}\dot{\gamma}_{ab} - \mathcal{L}, \\ &= \Pi^{ab}(-2\alpha K_{ab} + 2\bar{\nabla}_{(a}\beta_{b)}) - \left(\frac{1}{16\pi G}\alpha\sqrt{\gamma}(\bar{R} + K^{ab}K_{ab} - K^2 \right. \\ &\quad \left. + \frac{1}{\alpha}\bar{\nabla}^2\alpha + \frac{1}{8\pi G}\sqrt{\gamma}\bar{\nabla}_c(r^c K) \right), \\ &= -2\alpha\Pi^{ab}\left(\left(\frac{16\pi G}{\sqrt{\gamma}} \right) \left(\frac{1}{2}\gamma_{ab}\Pi - \Pi_{ab} \right) \right) + 2\Pi^{ab}\bar{\nabla}_{(a}\beta_{b)} - \frac{\alpha\sqrt{\gamma}}{16\pi G}\bar{R} \\ &\quad - \frac{\sqrt{\gamma}}{16\pi G}\bar{\nabla}^2\alpha - \alpha\left(\frac{16\pi G}{\sqrt{\gamma}} \right) \left(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2 \right) - \frac{1}{8\pi G}\sqrt{\gamma}\nabla_a(n^a K), \end{aligned}$$

⁷And note, that if we had chosen to try and write a Canonical theory based on the g_{ab} and the \dot{g}_{ab} as our variables, this program would have failed here, as we would not have been able to invert equation (4.1.17) in order to get equation (4.1.23), and therefore, we would not have been able to write down a Hamiltonian in terms of the g_{ab} and the Π^{ab}

$$\begin{aligned}
&= \alpha \left(\frac{16\pi G}{\sqrt{\gamma}} \right) \left(\Pi^{ab} \Pi_{ab} - \frac{1}{2} \Pi^2 \right) - \frac{\alpha \sqrt{\gamma}}{16\pi G} \bar{R} + 2\Pi^{ab} \bar{\nabla}_{(a} \beta_{b)} \\
&\quad - \frac{\sqrt{\gamma}}{16\pi G} \bar{\nabla}^2 \alpha - \frac{1}{8\pi G} \sqrt{\gamma} \nabla_a (n^a K) .
\end{aligned} \tag{4.1.28}$$

Which is the ADM Hamiltonian density, as found originally by Arnowitt, Deser and Misner [11]. A 3-dimensional integral of this quantity would then yield the ADM Hamiltonian. Note that it includes boundary terms, and that these boundary terms must be treated properly. In order to do this, one must be quite careful about the proper treatment of boundary *charges*. In particular, it can be shown that the the above Hamiltonian can lead to inconsistencies if just naïvely varied. The reader is referred to the 5 for some background on properly treating these boundary terms.

4.2 Constraints

Now, we take the variation of the Hamiltonian density in order to find equations of motion. Total divergences will be neglected in this section, but will be treated explicitly according to the perspective of Chapter 5⁸. Note that the above Hamiltonian must be modified by the addition of a series of surface charges in order for the overall variation to be zero.

Having said all of this, it is now easy to derive the constraint equations.

⁸In this Chapter, we will use what we know about the bulk equations of motion in order to pick the appropriate boundary terms. Therefore, it is not explicitly necessary to carefully and faithfully preserve the boundary terms dictated by the Hilbert action. Instead, we faithfully maintain the appropriate behaviour *in the bulk*, and then let that behaviour dictate what our boundary terms should be

The Hamiltonian constraint is derived by taking the variation of equation (4.1.28) with respect to α . Neglecting the boundary terms, the result is:

$$\begin{aligned}\frac{\delta \mathcal{H}}{\delta \alpha} = 0 &= \left(\frac{16\pi G}{\sqrt{\gamma}} \right) \left(\Pi^{ab} \Pi_{ab} - \frac{1}{2} \Pi^2 \right) - \frac{\sqrt{\gamma}}{16\pi G} \bar{R}, \\ &= - \left(\frac{\sqrt{\gamma}}{16\pi G} \right) (\bar{R} - K^{ab} K_{ab} + K^2) .\end{aligned}\tag{4.2.1}$$

If it is desired, we could have added a matter density to the original Lagrangian, which typically would have been a function of g^{ab} and g_{ab} , but not their derivatives. Had we done this, we would have gone through the same (3+1) decomposition and Legendre transform process with these degrees of freedom⁹. This would give a well-defined matter Hamiltonian \mathcal{H}_m , which we could add to the Hamiltonian given in equation (4.1.28). Making the definition $\rho = \left(\frac{1}{\sqrt{\gamma}} \right) \frac{\partial \mathcal{H}_m}{\partial \alpha}$ then makes equation (4.2.1) equal to:

$$\frac{\delta \mathcal{H}_{total}}{\delta \alpha} = 0 = \frac{\delta \mathcal{H}_{Einstein}}{\delta \alpha} + \frac{\delta \mathcal{H}_m}{\delta \alpha} = \sqrt{\gamma} \left[\rho - \left(\frac{1}{16\pi G} \right) (\bar{R} - K^{ab} K_{ab} + K^2) \right] .\tag{4.2.2}$$

Which is the Hamiltonian constraint, as seen in equation (3.1.23).

Similarly, we can derive the Momentum constraint, but taking the variation of equation (4.1.28) with respect to β_a . As we do in Chapter 5, we do a little bit of algebra gymnastics, and ultimately, we discard the boundary

⁹This procedure is traced in detail for a Klein-Gordon field coupled to an external gravitational field in Appendix (B)

term, in anticipation of dealing with it in Chapter 5. Since β_a only appears in a single term above, we only worry about that single term. Furthermore, since Π^{ab} is a symmetric tensor, the symmetrization over the derivative of β_a is omitted as irrelevant.

$$\begin{aligned}
\frac{\delta \mathcal{H}}{\delta \beta_b} = 0 &= \frac{\delta \mathcal{H}_{Einstein}}{\delta \beta_b} + \frac{\delta \mathcal{H}_m}{\delta \beta_b}, \\
&= \frac{\delta}{\delta \beta_b} (2\Pi^{ab} \bar{\nabla}_a \beta_b) + \frac{\delta \mathcal{H}_m}{\delta \beta_b}, \\
&= \frac{\delta}{\delta \beta_b} \left(\left(\frac{\sqrt{\gamma}}{\sqrt{\gamma}} \right) 2\Pi^{ab} \bar{\nabla}_a \beta_b \right) + \frac{\delta \mathcal{H}_m}{\delta \beta_b}, \\
&= \frac{\delta}{\delta \beta_b} \left(2\sqrt{\gamma} \left[\bar{\nabla}_a \left(\left(\frac{1}{\sqrt{\gamma}} \right) \Pi^{ab} \beta_b \right) - \beta_b \bar{\nabla}_a \left(\left(\frac{1}{\sqrt{\gamma}} \right) \Pi^{ab} \right) \right] \right) + \frac{\delta \mathcal{H}_m}{\delta \beta_b}, \\
&= -2\sqrt{\gamma} \bar{\nabla}_a \left(\left(\frac{1}{\sqrt{\gamma}} \right) \Pi^{ab} \right) + \frac{\delta \mathcal{H}_m}{\delta \beta_b}, \\
&= \frac{1}{8\pi G} \bar{\nabla}_a (K^{ab} - \gamma^{ab} K) - j^b.
\end{aligned} \tag{4.2.3}$$

Where we used equation (4.1.17) and made the definition $j^a = -\frac{\delta \mathcal{H}_m}{\delta \beta_a}$ in the last step. This equation is the Momentum Constraint, as seen in equation (3.1.26).

4.3 Evolution equations

Now, we follow the standard Hamiltonian procedure to derive equations for the canonical field variables γ_{ab} and Π^{ab} .¹⁰ Namely, after deriving the appropriate

¹⁰For simplicity, the matter fields are set to zero in this section. We have already derived the correct ADM equations with matter above in Chapter 3, and we are soon to be drowned

Hamiltonian density and solving for the appropriate constraints, we vary the phase space action

$$S = \int d^4x \left(\Pi^{ab} \dot{\gamma}_{ab} - \mathcal{H} \right) . \quad (4.3.1)$$

with respect to the canonical variables. This gives:

$$\begin{aligned} \delta S &= \int d^4x \left[(\delta \Pi^{ab}) \dot{\gamma}_{ab} + \Pi^{ab} \delta (\dot{\gamma}_{ab}) - \frac{\delta \mathcal{H}}{\delta \Pi^{ab}} \delta \Pi^{ab} - \frac{\delta \mathcal{H}}{\delta \gamma_{ab}} \delta \gamma_{ab} \right] , \\ &= \int d\tau \int d^3x \left[\delta \Pi^{ab} \left(\dot{\gamma}_{ab} - \frac{\delta \mathcal{H}}{\delta \Pi^{ab}} \right) + \frac{d}{dt} (\Pi^{ab} \delta \gamma_{ab}) - \delta \gamma_{ab} \left(\dot{\Pi}^{ab} + \frac{\delta \mathcal{H}}{\delta \gamma_{ab}} \right) \right] . \end{aligned} \quad (4.3.2)$$

The first and third terms in equation (4.3.2) are simply equivalent to the Hamiltonian equations of motion $\dot{\gamma}_{ab} = \frac{\delta \mathcal{H}}{\delta \Pi^{ab}}$ and $\dot{\Pi}^{ab} = -\frac{\delta \mathcal{H}}{\delta \gamma_{ab}}$. The second term is a total time derivative. We can therefore do the time integral first over this term, giving us $\int d^3x \Pi^{ab} \delta \gamma_{ab}$ over the initial and final timeslices. We, however, assume that our system is set up in such a way that it has a definite set of initial conditions, which therefore means that the value of γ_{ab} is held fixed on the initial and final time slices. Therefore, any variation over these regions must vanish identically. We can therefore drop the second term in (4.3.2). Therefore, deriving the evolution equations is as simple as varying the Hamiltonian with respect to the 3-metric and its canonical momentum, Π^{ab} . This will then give us equations for $\dot{\Pi}^{ab}$ and $\dot{\gamma}_{ab}$. Note, however, that this does

under a large number of terms. Adding matter back in doesn't massively increase the complexity of what follows, but it does enough that it is not worth our trouble here.

not directly give us an equation for \dot{K}_{ab} . However, an inspection of equation (4.1.17) shows that one can solve for \dot{K}^{ab} if one knows the values of $\dot{\Pi}^{ab}$ and $\dot{\gamma}_{ab}$ ¹¹. Having derived the equation for \dot{K}_{ab} in equation (3.2.17), we will not concern ourselves with performing this transformation here. It will just be noted that this transformation can be performed through a relatively direct, if laborious, computation.

First, we find the equation for $\dot{\gamma}_{ab}$, neglecting terms not involving Π^{ab}

$$\begin{aligned}
\dot{\gamma}_{ab} &= \frac{\delta \mathcal{H}}{\delta \Pi^{ab}}, \\
&= \frac{\delta}{\delta \Pi^{ab}} \left[\alpha \left(\frac{16\pi G}{\sqrt{\gamma}} \right) \left(\Pi^{ab} \Pi_{ab} - \frac{1}{2} \Pi^2 \right) + 2 \Pi^{ab} \bar{\nabla}_{(a} \beta_{b)} \right], \\
&= 2\alpha \left(\frac{16\pi G}{\sqrt{\gamma}} \right) \left(\Pi_{ab} - \frac{1}{2} \gamma_{ab} \Pi \right) + 2 \bar{\nabla}_{(a} \beta_{b)}, \\
&= -2\alpha K_{ab} + 2 \bar{\nabla}_{(a} \beta_{b)}.
\end{aligned} \tag{4.3.4}$$

Where we used equation (4.1.23) in order to convert the Π_{ab} terms into K_{ab} terms. Equation (4.3.4) is clearly the same equation as (3.2.5).

Before we derive the equation for $\dot{\Pi}^{ab}$, we derive a few properties of the

¹¹In accomplishing this, it is absolutely useful to use the identity derived from the definition of the inverse metric:

$$\begin{aligned}
\mathcal{L}_t \gamma_{ab} &= \mathcal{L}_t (\gamma_{ac} \gamma^{cd} \gamma_{db}), \\
&= (\mathcal{L}_t \gamma_{ac}) \gamma^{cd} \gamma_{db} + \gamma_{ac} (\mathcal{L}_t \gamma^{cd}) \gamma_{db} + \gamma_{ac} \gamma^{cd} \mathcal{L}_t \gamma_{db}, \\
&= \mathcal{L}_t \gamma_{ab} + \gamma_{ac} \gamma_{db} \mathcal{L}_t \gamma^{cd} + \mathcal{L}_t \gamma_{ba}, \\
-\mathcal{L}_t \gamma_{ab} &= \gamma_{ac} \gamma_{db} \mathcal{L}_t \gamma^{cd}.
\end{aligned} \tag{4.3.3}$$

variation of quantities with respect to $\delta\gamma_{ab}$. First, we tackle the determinant of the 3-metric ¹²:

$$\begin{aligned}
\frac{\delta}{\delta\gamma_{ab}}\gamma &= \frac{\delta}{\delta\gamma_{ab}} \left(\frac{1}{3!} \gamma_{mr} \gamma_{ns} \gamma_{kt} \epsilon^{mnk} \epsilon^{rst} \right) , \\
&= \frac{1}{2!} \gamma_{ns} \gamma_{kt} \epsilon^{ank} \epsilon^{bst} , \\
&= \gamma \gamma^{ab} \dots
\end{aligned} \tag{4.3.5}$$

Where the last step in (4.3.5) involved recognizing two things. First, that the matrix $\gamma_{ns} \gamma_{kt} \epsilon^{ank} \epsilon^{bst}$ is constructed entirely from the metric and the Levi-Civita symbol, and thus, can depend only on these two things. Second, that if one were to multiply $\gamma_{ns} \gamma_{kt} \epsilon^{ank} \epsilon^{bst}$ by γ_{ab} , the answer would be three times γ . Therefore, one can conclude the answer on the last line of (4.3.5). It is also not too hard to work out the variation of γ^{ab} with respect to γ_{ab} :

$$\begin{aligned}
\frac{\delta\gamma^{mn}}{\delta\gamma_{ab}} &= \frac{\delta}{\delta\gamma_{ab}} (\gamma^{mr} \gamma_{rs} \gamma^{sn}) , \\
&= \left(\frac{\delta\gamma^{mr}}{\delta\gamma_{ab}} \right) \gamma_{rs} \gamma^{sn} + \gamma^{mr} \left(\frac{\delta\gamma_{rs}}{\delta\gamma_{ab}} \right) \gamma^{sn} + \gamma^{mr} \gamma_{rs} \left(\frac{\delta\gamma^{sn}}{\delta\gamma_{ab}} \right) , \\
&= 2 \left(\frac{\delta\gamma^{mn}}{\delta\gamma_{ab}} \right) + \gamma^{mr} \gamma^{sn} \delta_{(r}^a \delta_s)^b , \\
- \left(\frac{\delta\gamma^{mn}}{\delta\gamma_{ab}} \right) &= \gamma^{m(a} \gamma^{b)n} .
\end{aligned} \tag{4.3.6}$$

¹²This derivation works in any dimension, not just three—in d dimensions, the Levi-Civita symbol has d indices, requiring d copies of the metric tensor, and making the factorial term $\frac{1}{d!}$. Then, the rest of this proof follows in exactly the same way as this one does.

And we can use equation (4.3.6) to easily change the variable of the variation between the metric and its inverse when it is convenient to do so—in particular, when taking the variation of the Ricci scalar with respect to the 3-metric.

Now, we embark upon taking the variation of the Hamiltonian (4.1.28) with respect to the 3-metric¹³:

$$\begin{aligned}
\delta \mathbf{H}|_\gamma &= \delta|_\gamma \int d^3x \left[\frac{16\pi G}{\sqrt{\gamma}} \alpha \left(\Pi^{ab} \Pi_{ab} - \frac{1}{2} \Pi^2 \right) - \frac{\alpha \sqrt{\gamma}}{16\pi G} \bar{R} \right. \\
&\quad \left. + 2\Pi^{ab} \bar{\nabla}_a \beta_b - \frac{\sqrt{\gamma}}{16\pi G} \bar{\nabla}^2 \alpha - \frac{1}{8\pi G} \sqrt{\gamma} \nabla_a (n^a K) \right], \\
&= \int d^3x \left[-\frac{8\pi G}{\sqrt{\gamma}} \gamma^{ab} \alpha \delta \gamma_{ab} \left(\Pi^{cd} \Pi_{cd} - \frac{1}{2} \Pi^2 \right) - \frac{\sqrt{\gamma}}{32\pi G} \gamma^{ab} \delta \gamma_{ab} \bar{\nabla}^2 \alpha \right. \\
&\quad \left. + \frac{32\pi G}{\sqrt{\gamma}} \alpha \delta \gamma_{ab} \left(\Pi^a{}_c \Pi^{cb} - \frac{1}{2} \Pi^{ab} \Pi \right) - \frac{\sqrt{\gamma}}{16\pi G} \delta \gamma_{ab} \bar{\nabla}^a \bar{\nabla}^b \alpha \right. \\
&\quad \left. - \frac{\alpha \sqrt{\gamma}}{32\pi G} \bar{R} \gamma^{ab} \delta \gamma_{ab} - \frac{\alpha \sqrt{\gamma}}{16\pi G} \left(\bar{R}^{ab} \delta \gamma_{ab} + \gamma^{ab} \delta \bar{R}_{ab}|_\gamma \right) - 2\beta_c \Pi^{ab} \delta \bar{\Gamma}_{ab}{}^c \right].
\end{aligned} \tag{4.3.7}$$

We will deal with the rest of these terms in their due course, but let us first consider the last term involving the variation of the Ricci tensor in (4.3.7).

¹³Note that the variation of the $\Pi^{ab} \nabla_a \beta_b$ term with respect to γ_{ab} depends only on the variation of the 3-Christoffel symbol with respect to the metric, due to the fact that the canonical form of the Π^{ab} is considered to be the raised one, and the canonical form of β_a is the one that has the lowered index. Thus, there is no dependence on γ_{ab} in this term.

4.3.1 The variation of the Ricci tensor

Tackling this term, we start with:

$$\delta \mathcal{H}_{Ricci} = - \int d^3x \frac{1}{16\pi G} (\alpha \sqrt{\gamma} \gamma^{ab} \delta \bar{R}_{ab}) . \quad (4.3.8)$$

Following [11], define $\kappa \equiv \frac{1}{16\pi G}$ for simplicity. Our goal is then to compute this variation of \bar{R}_{ab} in terms of the variation of γ_{ab} . To do this, we first expand \bar{R}_{ab} in terms of Christoffel symbols. We then remember that since the difference of two connections is a tensor, therefore, the difference of two Christoffel symbols is also a tensor, making the variation of a Christoffel symbol a tensor. We can then turn partial derivatives into covariant derivatives relative to the central metric about which we take the variation ¹⁴:

$$\begin{aligned} \delta \mathcal{H}_{Ricci} &= - \int d^3x \kappa \alpha \sqrt{\gamma} \gamma^{ab} \delta (\partial_c \bar{\Gamma}_{ab}^c - \partial_a \bar{\Gamma}_{bc}^c + \bar{\Gamma}_{ab}^d \bar{\Gamma}_{dc}^c - \bar{\Gamma}_{ac}^d \bar{\Gamma}_{db}^c) , \\ &= - \int \sqrt{\gamma} d^3x \kappa \alpha \gamma^{ab} (\partial_c \delta \bar{\Gamma}_{ab}^c - \partial_a \delta \bar{\Gamma}_{bc}^c + \bar{\Gamma}_{dc}^c \delta \bar{\Gamma}_{ab}^d + \bar{\Gamma}_{ab}^d \delta \bar{\Gamma}_{dc}^c \\ &\quad - \bar{\Gamma}_{ac}^d \delta \bar{\Gamma}_{db}^c - \bar{\Gamma}_{db}^c \delta \bar{\Gamma}_{ac}^d) , \\ &= - \int \sqrt{\gamma} d^3x \kappa \alpha \gamma^{ab} (\bar{\nabla}_c \delta \bar{\Gamma}_{ab}^c + \bar{\Gamma}_{ca}^d \delta \bar{\Gamma}_{db}^c + \bar{\Gamma}_{cb}^d \delta \bar{\Gamma}_{ad}^c - \bar{\Gamma}_{dc}^d \delta \bar{\Gamma}_{ab}^c \\ &\quad - \bar{\nabla}_a \delta \bar{\Gamma}_{bc}^c - \bar{\Gamma}_{ab}^d \delta \bar{\Gamma}_{dc}^c + \bar{\Gamma}_{dc}^c \delta \bar{\Gamma}_{ab}^d + \bar{\Gamma}_{ab}^d \delta \bar{\Gamma}_{dc}^c \\ &\quad - \bar{\Gamma}_{ac}^d \delta \bar{\Gamma}_{db}^c - \bar{\Gamma}_{db}^c \delta \bar{\Gamma}_{ac}^d) , \\ &= - \int \sqrt{\gamma} d^3x \kappa \alpha \gamma^{ab} (\bar{\nabla}_c \delta \bar{\Gamma}_{ab}^c - \bar{\nabla}_a \delta \bar{\Gamma}_{bc}^c) , \end{aligned}$$

¹⁴also, note that variations are defined in terms of partial derivatives in function space. Therefore, they can be freely commuted through partial derivatives

$$= - \int \sqrt{\gamma} d^3x \kappa \alpha (\gamma^{ab} \delta^d_c - \gamma^{da} \delta^b_c) \bar{\nabla}_d \delta \bar{\Gamma}_{ab}^c. \quad (4.3.9)$$

So, we have now reduced this problem to evaluating the gradient of the variation of the Christoffel symbol in terms of the variation of the 3-metric. As with almost all variational problems, we proceed by integrating by parts¹⁵.

$$\begin{aligned} \delta \mathcal{H}_{Ricci} &= - \int \sqrt{\gamma} d^3x \left\{ \kappa \bar{\nabla}_d [\alpha (\gamma^{ab} \delta^d_c - \gamma^{ad} \delta^b_c) \delta \bar{\Gamma}_{ab}^c] \right. \\ &\quad \left. - \kappa \bar{\nabla}_d [\alpha (\gamma^{ab} \delta^d_c - \gamma^{ad} \delta^b_c)] \delta \Gamma_{ab}^c \right\}, \\ &= - \oint \sqrt{q} d^2x \kappa \alpha (\gamma^{ab} \bar{r}_c - \bar{r}^a \delta^b_c) \delta \bar{\Gamma}_{ab}^c \\ &\quad + \int \sqrt{\gamma} d^3x \kappa (\gamma^{ab} \bar{\nabla}_c \alpha - \delta^b_c \bar{\nabla}^a \alpha) \delta \bar{\Gamma}_{ab}^c, \end{aligned} \quad (4.3.10)$$

$$= \oint \sqrt{q} d^2x B^{ab}_c \delta \bar{\Gamma}_{ab}^c + \int \sqrt{\gamma} d^3x T^{ab}_c \delta \bar{\Gamma}_{ab}^c. \quad (4.3.11)$$

Where we make the definitions $B^{ab}_c \equiv \kappa \alpha (\frac{1}{2} \bar{r}^a \delta^b_c + \frac{1}{2} \bar{r}^b \delta^a_c - \gamma^{ab} \bar{r}_c)$ and $T^{ab}_c \equiv \kappa (\gamma^{ab} \bar{\nabla}_c \alpha - \frac{1}{2} \delta^b_c \bar{\nabla}^a \alpha - \frac{1}{2} \delta^a_c \bar{\nabla}^b \alpha)$ ¹⁶. Note that both objects are true tensors whose indices can be freely raised and lowered with γ_{ab} and its inverse. Now, our variational problem has been reduced to finding the variation of a three index tensor multiplied by $\delta \Gamma_{ab}^c$. In order to do this, we quickly work out a property relating the variation of the metric to the variation of its inverse:

¹⁵Note that if the factor of α were not present, we could simply pull $\bar{\nabla}_d$ through to the beginning of the equation and the integration by parts would produce nothing but a boundary term

¹⁶Since no allowed variation of the metric will induce torsion in the connexion, we know that $\delta \Gamma_{ab}^c$ is symmetric on a and b. Therefore, we can symmetrize on these indices of B^{ab}_c and T^{ab}_c

$$\begin{aligned}
\delta\gamma^{ab} &= \delta \left(\gamma^{ac} \gamma_{cd} \gamma^{db} \right) , \\
&= (\delta\gamma^{ac}) \gamma_{cd} \gamma^{db} + \gamma^{ac} (\delta\gamma_{cd}) \gamma^{db} + \gamma^{ac} \gamma_{cd} (\delta\gamma^{db}) , \\
&= 2\delta\gamma^{ab} + \gamma^{ac} \gamma^{db} \delta\gamma_{cd} , \\
\delta\gamma^{ab} &= -\gamma^{ac} \gamma^{db} \delta\gamma_{cd} .
\end{aligned} \tag{4.3.12}$$

Having this identity in hand, we now work out the general expression for a tensor multiplying a variation of a Christoffel symbol. Using the decomposition the the Christoffel symbol into metric components, and converting partial derivatives into covariant derivatives, we obtain the following:

$$\begin{aligned}
M^{ab}{}_c \delta\Gamma_{ab}{}^c &= \frac{1}{2} M^{ab}{}_c \delta \left[\gamma^{cd} (\partial_a \gamma_{bd} + \partial_b \gamma_{ad} - \partial_d \gamma_{ab}) \right] , \\
&= -\frac{1}{2} M^{ab}{}_c (\gamma^{ce} \gamma^{df} \delta\gamma_{ef}) (\partial_a \gamma_{bd} + \partial_b \gamma_{ad} - \partial_d \gamma_{ab}) \\
&\quad + \frac{1}{2} M^{ab}{}_c \gamma^{cd} (\partial_a \delta\gamma_{bd} + \partial_b \delta\gamma_{ad} - \partial_d \delta\gamma_{ab}) , \\
&= -M^{abe} \bar{\Gamma}_{ab}{}^f \delta\gamma_{ef} + \frac{1}{2} M^{abd} (\bar{\nabla}_a \delta\gamma_{bd} + \bar{\Gamma}_{ab}{}^c \delta\gamma_{cd} + \bar{\Gamma}_{ad}{}^c \delta\gamma_{bc} \\
&\quad + \bar{\nabla}_b \delta\gamma_{ad} + \bar{\Gamma}_{ba}{}^c \delta\gamma_{cd} + \bar{\Gamma}_{bd}{}^c \delta\gamma_{ac} - \bar{\nabla}_a \delta\gamma_{ab} - \bar{\Gamma}_{da}{}^c \delta\gamma_{cb} - \bar{\Gamma}_{db}{}^c \delta\gamma_{ac}) , \\
&= \frac{1}{2} M^{abc} (\bar{\nabla}_a \delta\gamma_{bc} + \bar{\nabla}_b \delta\gamma_{ac} - \bar{\nabla}_c \delta\gamma_{ab}) , \\
&= \frac{1}{2} (M^{abc} + M^{bac} - M^{cab}) \bar{\nabla}_a \delta\gamma_{bc} .
\end{aligned} \tag{4.3.13}$$

Using the above definitions and symmetrizing over the indices b and c,

since the multiplier is $\bar{\nabla}_a \gamma_{bc}$:

$$\begin{aligned}
B^{abc} + B^{bac} - B^{cab} &= \kappa \alpha \left(\frac{1}{2} \bar{r}^a \gamma^{bc} + \frac{1}{2} \bar{r}^b \gamma^{ac} - \gamma^{ab} \bar{r}^c + \frac{1}{2} \bar{r}^b \gamma^{ac} + \frac{1}{2} \bar{r}^a \gamma^{bc} \right. \\
&\quad \left. - \gamma^{ab} \bar{r}^c - \frac{1}{2} \bar{r}^c \gamma^{ab} - \frac{1}{2} \bar{r}^a \gamma^{bc} + \gamma^{ca} \bar{r}^b \right), \\
&= \kappa \alpha \left(\bar{r}^a \gamma^{bc} + 2 \bar{r}^b \gamma^{ac} - \frac{5}{2} \bar{r}^c \gamma^{ab} \right), \\
(B^{abc} + B^{bac} - B^{cab})_{sym,bc} &= \kappa \alpha \left(\bar{r}^a \gamma^{bc} - \gamma^{a(b} \bar{r}^{c)} \right), \tag{4.3.14}
\end{aligned}$$

$$\begin{aligned}
T^{abc} + T^{bac} - T^{cab} &= \kappa \left(\gamma^{ab} \bar{\nabla}^c \alpha - \frac{1}{2} \gamma^{bc} \bar{\nabla}^a \alpha - \frac{1}{2} \gamma^{ac} \bar{\nabla}^b \alpha + \gamma^{ba} \bar{\nabla}^c \alpha \right. \\
&\quad \left. - \frac{1}{2} \gamma^{ac} \bar{\nabla}^b \alpha - \frac{1}{2} \gamma^{bc} \bar{\nabla}^a \alpha - \gamma^{cb} \bar{\nabla}^a \alpha \right. \\
&\quad \left. + \frac{1}{2} \gamma^{ab} \bar{\nabla}^c \alpha + \frac{1}{2} \gamma^{cb} \bar{\nabla}^a \alpha \right), \\
&= \kappa \left(\frac{5}{2} \gamma^{ab} \bar{\nabla}^c \alpha - \gamma^{bc} \bar{\nabla}^a \alpha - 2 \gamma^{ac} \bar{\nabla}^b \alpha \right), \\
(T^{abc} + T^{bac} - T^{cab})_{sym,bc} &= \kappa \left(\gamma^{a(b} \bar{\nabla}^{c)} \alpha - \gamma^{bc} \bar{\nabla}^a \alpha \right). \tag{4.3.15}
\end{aligned}$$

Thus, equations (4.3.14) and (4.3.15) give us the final expressions for the terms multiplying $\bar{\nabla}_a \gamma_{bc}$ in the variation in equation (4.3.11), since equation (4.3.13) tells us how to convert the variation of a Christoffel symbol into the gradient of the variation of the 3-metric. Putting all of these equations together, and then integrating by parts¹⁷, we obtain:

¹⁷note that we automatically apply Gauss's theorem to the divergence term originating from the bulk term.

$$\begin{aligned}
\delta \mathcal{H}_{Ricci} &= \oint \sqrt{q} d^2 x \frac{1}{2} \kappa \alpha \left(\bar{r}^a \gamma^{bc} - \gamma^{a(b} \bar{r}^{c)} \right) \bar{\nabla}_a \delta \gamma_{bc} \\
&+ \int \sqrt{\gamma} d^3 x \frac{1}{2} \kappa \left(\gamma^{a(b} \bar{\nabla}^{c)} \alpha - \gamma^{bc} \bar{\nabla}^a \alpha \right) \bar{\nabla}_a \delta \gamma_{bc}, \\
&= \oint \sqrt{q} d^2 x \left\{ \frac{\kappa}{2} \bar{\nabla}_a \left[\alpha \left(\bar{r}^a \gamma^{bc} - \gamma^{a(b} \bar{r}^{c)} \right) \delta \gamma_{bc} \right] \right. \\
&\quad \left. - \frac{\kappa}{2} \delta \gamma_{bc} \bar{\nabla}_a \left[\alpha \left(\bar{r}^a \gamma^{bc} - \gamma^{a(b} \bar{r}^{c)} \right) \right] + \frac{\kappa}{2} \bar{r}_a \left(\gamma^{a(b} \bar{\nabla}^{c)} \alpha - \gamma^{bc} \bar{\nabla}^a \alpha \right) \delta \gamma_{bc} \right\} \\
&+ \int \sqrt{\gamma} d^3 x \frac{\kappa}{2} \bar{\nabla}_a \left(\gamma^{bc} \bar{\nabla}^a \alpha - \gamma^{a(b} \bar{\nabla}^{c)} \alpha \right) \delta \gamma_{bc}. \\
\\
&= \oint \sqrt{q} d^2 x \frac{\kappa}{2} \left\{ \nabla_a \left[\alpha \left(\bar{r}^a \gamma^{bc} - \gamma^{a(b} \bar{r}^{c)} \right) \delta \gamma_{bc} \right] \right. \\
&\quad \left. + \delta \gamma_{bc} \left[\alpha \left(\gamma^{bc} \bar{\nabla}_a \bar{r}^a - \bar{\nabla}^{(b} \bar{r}^{c)} \right) + 2 \bar{r}^{(b} \bar{\nabla}^{c)} \alpha - 2 \gamma^{bc} \bar{r}^a \bar{\nabla}_a \alpha \right] \right\} \\
&+ \int \sqrt{\gamma} d^3 x \frac{\kappa}{2} \left(\gamma^{bc} \bar{\nabla}^a \bar{\nabla}_a \alpha - \bar{\nabla}^{(b} \bar{\nabla}^{c)} \alpha \right) \delta \gamma_{bc}. \tag{4.3.16}
\end{aligned}$$

Now, equation (4.3.16) nearly has the form that we want. Most of the terms are direct multipliers of the variation of the 3-metric γ_{bc} . We are left, however, with a 3-divergence to be integrated over the boundary. The naïve thought would be to simply eliminate this term using Gauss's theorem, since it should be convertible to an integral over the boundary of the boundary, and a well-known theorem of topology tells us that the boundary of the boundary is zero for any manifold.

Finally, remembering our promise to wait until Chapter 5 to deal with the boundary terms, we drop everything in (4.3.16) except for the bulk term,

and our final answer is:

$$\int d^3x \gamma^{ab} \delta \bar{R}_{ab}|_\gamma = \int d^3x \sqrt{\gamma} \frac{\kappa}{2} (\gamma^{ab} \bar{\nabla}^2 \alpha - \bar{\nabla}^a \bar{\nabla}^b \alpha) \delta \gamma_{ab}. \quad (4.3.17)$$

4.3.2 The equation for $\dot{\Pi}^{ab}$

Now, having gone through the work of that extensive aside, we are ready to return to the equation for $\dot{\Pi}^{ab}$ that we obtained from the ADM Hamiltonian. Substituting (4.3.17) into (4.3.7), and using (4.3.13) to deal with the term in (4.3.7) involving the shift vector¹⁸, we get:

$$\begin{aligned} \delta \mathbf{H}|_\gamma &= \int d^3x \left[-\frac{8\pi G}{\sqrt{\gamma}} \gamma^{ab} \alpha \delta \gamma_{ab} \left(\Pi^{cd} \Pi_{cd} - \frac{1}{2} \Pi^2 \right) - \frac{\sqrt{\gamma}}{32\pi G} \gamma^{ab} \delta \gamma_{ab} \bar{\nabla}^2 \alpha \right. \\ &\quad + \frac{32\pi G}{\sqrt{\gamma}} \alpha \delta \gamma_{ab} \left(\Pi^a{}_c \Pi^{cb} - \frac{1}{2} \Pi^{ab} \Pi \right) - \frac{\sqrt{\gamma}}{16\pi G} \delta \gamma_{ab} \bar{\nabla}^a \bar{\nabla}^b \alpha \\ &\quad - \frac{\alpha \sqrt{\gamma}}{32\pi G} \bar{R} \gamma^{ab} \delta \gamma_{ab} - \frac{\alpha \sqrt{\gamma}}{16\pi G} \left(\bar{R}^{ab} \delta \gamma_{ab} + \frac{1}{2} (\gamma^{ab} \bar{\nabla}^2 \alpha - \bar{\nabla}^a \bar{\nabla}^b \alpha) \delta \gamma_{ab} \right) \\ &\quad \left. - \left(\frac{\sqrt{\gamma}}{\sqrt{\gamma}} \right) (\Pi^{ab} \beta^c - \Pi^{bc} \beta^a - \Pi^{ca} \beta^b) \bar{\nabla}_c \delta \gamma_{ab} \right], \\ &= \int d^3x \delta \gamma_{ab} \left[\frac{8\pi G \alpha}{\sqrt{\gamma}} \left(\frac{1}{2} \gamma^{ab} \Pi^2 - \gamma^{ab} \Pi^{cd} \Pi_{cd} + 4 \Pi^a{}_c \Pi^{cb} - 2 \Pi^{ab} \Pi \right) \right. \\ &\quad + \frac{\sqrt{\gamma}}{16\pi G} \left(-\alpha \bar{R}^{ab} - \alpha \frac{1}{2} \gamma^{ab} \bar{R} - \bar{\nabla}^a \bar{\nabla}^b \alpha \right) \\ &\quad \left. + \sqrt{\gamma} \delta \gamma_{ab} \bar{\nabla}_c \left\{ \frac{1}{\sqrt{\gamma}} (\Pi^{ab} \beta^c - \Pi^{ac} \beta^b - \Pi^{bc} \beta^a) \right\} \right]. \quad (4.3.18) \end{aligned}$$

¹⁸Once again, neglecting boundary terms, as they will be treated with in Chapter 5

Finally for all of that effort, we get our final solution, after referring to our original phase space action variation in (4.3.2):

$$\begin{aligned}
\dot{\Pi}^{ab} &= -\frac{\delta \mathcal{H}|_\gamma}{\delta \gamma_{ab}}, \\
&= \frac{8\pi G \alpha}{\sqrt{\gamma}} \left(-\frac{1}{2} \gamma^{ab} \Pi^2 + \gamma^{ab} \Pi^{cd} \Pi_{cd} - 4 \Pi^a{}_c \Pi^{cb} + 2 \Pi^{ab} \Pi \right) \\
&\quad + \frac{\sqrt{\gamma}}{16\pi G} \left(\alpha \bar{R}^{ab} + \alpha \frac{1}{2} \gamma^{ab} \bar{R} + \bar{\nabla}^a \bar{\nabla}^b \alpha \right) \\
&\quad + \sqrt{\gamma} \delta \gamma_{ab} \bar{\nabla}_c \left[\frac{1}{\sqrt{\gamma}} \left(-\Pi^{ab} \beta^c + \Pi^{ac} \beta^b + \Pi^{bc} \beta^a \right) \right]. \tag{4.3.19}
\end{aligned}$$

Equation (4.3.19) can be shown, through a laborious calculation involving (4.1.23) and (4.3.4), to be equivalent to (3.2.17). We have now completed our derivation of the 3+1 Einstein equations in the bulk using the Hamiltonian formalism. Now, we move on and work with the boundary terms in the next Chapter.

Chapter 5

Boundary Charges and Counterterms

5.1 Gauss's Theorem

Now, let us consider the way to extract information about boundary charges out of the ADM formalism. For our current purposes, we will not concern ourselves with the $\int \sqrt{\gamma} d^3x K$ terms over the initial and final slices in (4.1.28). Instead, we focus on the term that has the form $\int d\tau \int d^3x \sqrt{\gamma} \bar{\nabla}^2 \alpha$. Before we progress, we will restate Gauss's theorem as described in Hawking and Ellis [12]:

First, consider the integral of the following quantity over an arbitrary region \mathcal{R} of a n -dimensional manifold \mathcal{M} , endowed with metric tensor g_{ab} , and metric-compatible connection ∇_a :

$$\int_{\mathcal{R}} d^n x \sqrt{|g|} \nabla_a V^a. \quad (5.1.1)$$

Since the exterior derivative acting on the volume element necessarily yields zero (there are no totally antisymmetric $(n+1)$ dimensional objects over a n dimensional manifold), we can apply integration by parts to the above term. This reduces the dimension of the volume element by one, and gives us an integral over the boundary of \mathcal{R} :

$$\int_{\mathcal{R}} d^n x \sqrt{|g|} \nabla_a V^a = \int_{\partial \mathcal{R}} d^{n-1} x \sqrt{|\gamma|} r_a V^a. \quad (5.1.2)$$

Where the integral in (5.1.2) is over the boundary of \mathcal{R} , and r_a is the unit “outward” normal to this boundary. If one prefers to think of n dimensional quantities instead of $n-1$ dimensional quantities, one can define the boundary of \mathcal{R} as the zero of a scalar function ρ ¹. We can then define “ $r_a'' = \nabla_a \rho$ ”.

“ r_a'' ”, however, differs from r_a merely by a factor of $\frac{1}{\sqrt{\nabla_a \rho \nabla^a \rho}}$. A simple computation then shows that $(\sqrt{\nabla_a \rho \nabla^a \rho})(\sqrt{|\gamma|}) = \sqrt{|g|}$. Therefore, if one would prefer, (5.1.2) could be rewritten as:

$$\int_{\mathcal{R}} d^n x \sqrt{|g|} \nabla_a V^a = \int_{\partial \mathcal{R}} d^{n-1} x \sqrt{|g|} r_a'' V^a. \quad (5.1.3)$$

For a great many applications, this conversion greatly simplifies the calculation of an integral, as the domain of dependence for the problem has been reduced from the entire bulk of the problem to merely its boundary.

¹For most cases, we would take this to be something supremely simple, like $\rho = r - r_0$, but it is not absolutely necessary to do so.

Furthermore, since the boundary of the boundary is zero by a famous theorem from topology, we can immediately conclude, that the right hand side of (5.1.2) vanishes if we are able to repeat this procedure. We should not, however, take this to mean that the integral of a n -dimensional Laplacian over the bulk always vanishes. First, we will see a simple counterexample to show that naïve double application of Gauss's Theorem results in nonsense, and then it will be explained why this is the case. First, consider the differential form of Gauss's Law:

$$\nabla_a E^a = \frac{1}{\epsilon_0} \rho. \quad (5.1.4)$$

It is a simple matter to integrate this equation over space on both sides, and then apply Gauss's theorem in order to obtain the integral form of Gauss's Law (for clarity, we absorb the metric determinant factors into the volume elements):

$$\int_{\mathcal{M}} d^3x \nabla_a E^a = \int_{\partial\mathcal{M}} d^2x r_a E^a = \int d^3x \frac{\rho}{\epsilon_0} = \frac{Q_{inc}}{\epsilon_0}. \quad (5.1.5)$$

Now, one might then recall that the Electric field is merely -1 times the gradient of the electrostatic potential ϕ , and then replace E^a with $-\nabla^a \phi$ in (5.1.5):

$$\frac{Q_{inc}}{\epsilon_0} = - \int_{\partial\mathcal{M}} d^2x r_a \nabla^a \phi,$$

$$= \int_{\partial\mathcal{M}} d^2x \phi \nabla_a r^a - \int_{\partial\mathcal{M}} d^2x \nabla_a (\phi r^a), \quad (5.1.6)$$

$$\text{“} = \text{”} \int_{\partial\mathcal{M}} d^2x \phi \nabla_a r^a - \int_{\partial\mathcal{M}} d^1x s_a \phi r^a, \quad (5.1.7)$$

$$\text{“} = \text{”} \int_{\partial\mathcal{M}} d^2x \phi \nabla_a r^a. \quad (5.1.8)$$

Where we used the metric compatibility of ∇_a to raise and lower indices both inside and outside the gradient. A computation of (5.1.8) for even the simplest examples from Electrostatics will show that (5.1.8) is nonsense, however. For instance, if we simply consider the static point charge field from Coloumb’s law, we have $\phi = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$, while for r^a normal to a sphere in Euclidean space, we have $\nabla_a r^a = \frac{2}{r}$. Therefore, the right hand side of (5.1.8) comes out to $\int_0^{2\pi} \int_0^\pi d\phi d\theta r^2 \sin(\theta) \frac{q}{2\pi\epsilon_0 r^2}$, which then causes (5.1.8) to yield $\frac{Q_{inc}}{\epsilon_0} = \frac{2q}{\epsilon_0}$, which is clearly incorrect.

What is the origin of this error, however? It turns out that if you compute the second integral in (5.1.6), you will find that it has value $\frac{q}{\epsilon_0}$, which accounts for the discrepancy between the left and right hand sides of (5.1.8). But, then, what of our application of Gauss’s Theorem here? It must have been incorrect, as it assigned a zero value to something that was not zero. This, in fact, was our problem, and this is the reason why care must be taken with Gauss’s Theorem. The reason why we cannot apply Gauss’s theorem to (5.1.6) lies in the *type* of derivative operator that appears in (5.1.6)—that ∇_a is the derivative operator that is compatible with the *bulk* 3-metric, but not necessarily with the *boundary* two metric—in fact, since areas increase as we increase the value of r , the boundary two metric *cannot* be compatible with

∇_a —the bulk derivative operator should be able to record information about the increase of areas. Gauss’s Theorem would only be applicable to (5.1.6) if the derivative operator appearing there were the two connection compatible with the two metric. Therefore, care must be taken at all times with these operators, lest unwanted errors creep into calculations.

5.2 Boundary Charges in Maxwell Theory

Now, let us extend this calculation to the full Maxwell Hamiltonian. We start with the Maxwell Lagrangian with arbitrary external current j^a :

$$\mathcal{L} = \frac{1}{4} F^{ab} F_{ab} + j^a A_a , . \quad (5.2.1)$$

here, as is usual, $F_{ab} \equiv \partial_a A_b - \partial_b A_a$, and indices are raised and lowered using the Minkowski metric η_{ab} . Due to the antisymmetry of F_{ab} , it should be clear that there are no time derivatives of A_t appearing in this action. Therefore, we should treat A_t as a Lagrange multiplier here. Now, we will 3+1 split the Maxwell Lagrangian along surfaces of constant t , defining $A_a = (\phi, A_i)$ and $j^a = (\rho, j^i)$. We wish to define a Hamiltonian from the given Lagrangian, using A_i and its canonical momentum Π^i as canonical variables. First, we solve for Π^i :

$$\Pi^i \equiv \frac{\delta \mathcal{L}}{\delta \dot{A}_i} = \frac{1}{2} F^{ab} \frac{\delta}{\delta(\partial_t A_i)} F_{ab} = F^{ti} = \partial^t A^i - \partial^i A^t . \quad (5.2.2)$$

So, now, the Maxwell Lagrangian can be rewritten as:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{4} F^{ab} F_{ab} + j^a A_a, \\
&= \frac{1}{2} F^{ti} F_{ti} + \frac{1}{4} F^{ij} F_{ij} + j^t A_t + j^i A_i, \\
&= -\frac{1}{2} \Pi^i \Pi_i + \frac{1}{4} F^{ij} F_{ij} + \rho \phi + j^i A_i.
\end{aligned} \tag{5.2.3}$$

Now, we perform the Legendre transformation to arrive at the Maxwell Hamiltonian density:

$$\begin{aligned}
\mathcal{H} &= \Pi^i \dot{A}_i - \mathcal{L}, \\
&= \Pi^i (-\Pi_i + \partial_i \phi) - \left(-\frac{1}{2} \Pi^i \Pi_i + \frac{1}{4} F^{ij} F_{ij} + \rho \phi + j^i A_i \right), \\
&= -\frac{1}{2} \Pi^i \Pi_i - \frac{1}{4} F^{ij} F_{ij} + \Pi^i \partial_i \phi - \rho \phi - j^i A_i.
\end{aligned} \tag{5.2.4}$$

Now, let us take a variation of the phase space action with respect to ϕ .

$$\begin{aligned}
\delta|_\phi \mathcal{H} &= \int d^4x \Pi^i \partial_i \delta\phi - \rho \delta\phi, \\
&= \int dt \int d^3x \partial_i (\Pi^i \phi) - \delta\phi \partial_i \Pi^i - \rho \delta\phi, \\
&= \int dt \left[\oint d^2x \bar{r}_i \Pi^i \delta\phi - \int d^3x (\rho + \partial_i \Pi^i) \delta\phi \right].
\end{aligned} \tag{5.2.5}$$

the term under the $\int d^3x$ is, upon realization that $\Pi^i = -E^i$, easily recognizable as Gauss's Law. The boundary term, however, is the integral form of Gauss's Law, only without an appropriate charge term to balance it. Furthermore, the value of ϕ in this formalism is pure gauge—there is no abstract reason it has to take any value at all. Therefore, the presence of the boundary term gives us no reason at all to believe that the action derived from (5.2.4) is stationary with respect to variations of ϕ .

Therefore, in this theory, the behaviour of the bulk can create problems with the maintenance of a stationary action, by the production of gauge-dependent boundary terms. This behaviour, however, can be controlled.

Consider an ansatz to resolve this inconsistency—we define the Hamiltonian as the 4-dimensional integral of the Hamiltonian density plus a boundary term dependent only on ϕ and an as-yet undetermined constant:

$$H = \int d^3x \left(-\frac{1}{2} \Pi^i \Pi_i - \frac{1}{4} F^{ij} F_{ij} + \Pi^i \partial_i \phi - \rho \phi - j^i A_i \right) , \\ + \oint d^2x Q \phi + \oint d^2x J^i A_i . \quad (5.2.6)$$

The variation of this Hamiltonian with respect to A_i and Π^i can be shown, relatively easily, to give Faraday's law and the Ampère-Maxwell law².

²Since there is a gauge invariance built into F_{ij} , a similar boundary term to the one described above is generated by the variation with respect to A_i . It is dealt with in the same way as the variation with respect to ϕ , by the addition of a set of constants J^i that are determined in order to cancel the boundary integrals. Since this treatment isn't difficult, and is near-identical in form to the boundary terms arising from the variation of ϕ , these terms are not explicitly treated here

$\nabla_i B^i = 0$ is given by the fact that F_{ij} is the exterior derivative of A_i and the second exterior derivative of any quantity is zero. We therefore expect the variation of the Hamiltonian with respect to the only remaining variable, ϕ , to give Gauss's law:

$$\begin{aligned}
\frac{\delta H}{\delta \phi} &= \frac{\delta}{\delta \phi} \left[\int d^3x (\Pi^i \partial_i \phi - \rho \phi) + Q \oint d^2x (\phi) \right] , \\
&= \frac{\delta}{\delta \phi} \left[\int d^3x (\partial_i (\Pi^i \phi) - \phi \partial_i \Pi^i - \rho \phi) + \oint d^2x (Q \phi) \right] , \\
&= \frac{\delta}{\delta \phi} \left[\int d^3x (-\phi \partial_i \Pi^i - \rho \phi) + \oint d^2x (Q \phi + r_i \Pi^i \phi) \right] \\
&= - \int d^3x (\partial_i \Pi^i + \rho) + \oint d^2x (Q + r_i \Pi^i) , \\
&= \int d^3x (\rho - \bar{\nabla}_i E^i) + \oint d^2x (Q - r_i E^i) . \tag{5.2.7}
\end{aligned}$$

Since $F^{ti} = \Pi^i = -E^i$, we recognize the integrand in the first term above as simply being the electric field for the given charge configuration. Since this equation has no time derivatives, we can therefore consider this first term as a constraint upon the equations of motion given by varying the Hamiltonian with respect to A_i and Π^i . Once we have solved the appropriate equations of motion, we can then evaluate the second term in the boundary integral in equation (5.2.7), and then assign the value to Q that cancels that boundary term. This process is not unlike the renormalization process used in quantum field theory—where certain terms are computed using prescribed Hamiltonian dynamics, and then appropriate counterterms are assigned the

appropriate values at the end of the calculation in order to yield consistent equations of motion.

This aside being taken, we now discuss the topic of boundary charges in the ADM formulation.

5.2.1 ADM Formulation

Now, consider the ADM Hamiltonian:

$$\begin{aligned} \mathbf{H} = & \frac{1}{16\pi G} \int d^3x \left[-\alpha\sqrt{\gamma}\bar{R} + \frac{\alpha}{\sqrt{\gamma}}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2) \right. \\ & \left. -\sqrt{\gamma}(\bar{\nabla}^2\alpha + \alpha\nabla_a(n^a K)) + 2\Pi^{ab}\bar{\nabla}_a\beta_b \right] + \mathbf{H}_m. \end{aligned} \quad (5.2.8)$$

The $\nabla_a(n^a K)$ term is acted on with a 4-dimensional divergence operator. Therefore, it contributed a quantity of action $\int \sqrt{\gamma}d^3x K$ over the initial and final time slices, but nowhere else, provided that the spacelike normal to the outer boundary is normal to the timelike normal³. We therefore do not concern ourselves with this term. Now, let us do some massaging to the second to last term, involving the shift vector β_a :

$$\frac{1}{8\pi G} \int d^3x \Pi^{ab}\bar{\nabla}_a\beta_b = \frac{1}{8\pi G} \int d^3x \frac{\sqrt{\gamma}}{\sqrt{\gamma}} \Pi^{ab}\bar{\nabla}_a\beta_b,$$

³Note that Appendix D shows that the inner product between the radial vector and the timelike normal to the 3+1 slices is given by a boost parameter at infinity. We can therefore appropriately choose a slicing that gives a zero boost parameter between the two at infinity. In the asymptotically flat case, this is equivalent to the condition that $\beta^a \frac{1}{r}$

$$= \frac{1}{8\pi G} \int d^3x \sqrt{\gamma} \left(\bar{\nabla}_a \left(\frac{1}{\sqrt{\gamma}} \beta_b \Pi^{ab} \right) - \beta_b \bar{\nabla}_a \left(\frac{1}{\sqrt{\gamma}} \Pi^{ab} \right) \right).$$

We group the second term above along with the first and second terms in (5.2.8) and the matter Hamiltonian into the generic term \mathbf{H}_b , and refer to them as the bulk Hamiltonian density. Substituting the above result into (5.2.8) gives us:

$$\mathbf{H} = \mathbf{H}_b + \frac{1}{8\pi G} \int d^3x \sqrt{\gamma} \bar{\nabla}_a \left(\frac{1}{\sqrt{\gamma}} \beta_b \Pi^{ab} \right) - \frac{1}{16\pi G} \int d^3x \sqrt{\gamma} \bar{\nabla}^2 \alpha. \quad (5.2.9)$$

Now, since we have two integrals of divergences over the bulk manifold, we can convert these integrals into integrals over the boundary of the spacelike slice:

$$\mathbf{H} = \mathbf{H}_b + \frac{1}{8\pi G} \int d^2x \sqrt{q} r_a \beta_b (\gamma^{ab} K - K^{ab}) - \frac{1}{16\pi G} \int d^2x \sqrt{q} r_a \bar{\nabla}^a \alpha. \quad (5.2.10)$$

Now, we might be tempted to deal with the $\bar{\nabla}^a \alpha$ term by pulling the $\bar{\nabla}^a$ through, integrating by parts, reapplying Gauss's Theorem, and relying on the vanishing of the boundary of the boundary. Hopefully, the example from Equation (5.1.2) provides sufficient intuition to show that, since we have a $\bar{\nabla}$ and not a $\hat{\nabla}$ appearing in (5.2.10), this path will lead us inexorably to nonsense. Instead, we must consider the Boundary Hamiltonian to have the form provided above, and to take this seriously as we take our variations, which

we will do below.

5.2.2 Variation of the Boundary Hamiltonian

When we take the variation of the Hamiltonian with respect to the lapse function, we obtain:

$$\begin{aligned}\frac{\partial \mathbf{H}}{\partial \alpha} &= 0 = \frac{\partial \mathbf{H}_b}{\partial \alpha} + \frac{1}{16\pi G} \int d^2x \sqrt{q} r^a \frac{\partial \bar{\nabla}_a \alpha}{\partial \alpha}, \\ &= \mathcal{C} - \frac{1}{16\pi G} \int d^2x \bar{\nabla}_a (\sqrt{q} r^a).\end{aligned}\tag{5.2.11}$$

Taking $\frac{\partial \mathbf{H}}{\partial \alpha}$ simply gives the Hamiltonian constraint

$$\int d^3x \sqrt{\gamma} \left(\rho - \frac{1}{16\pi G} (\bar{R} - K^{ab} K_{ab} + K^2) \right)$$

where $\rho \equiv \frac{\partial \mathcal{H}_{matter}}{\partial \alpha}$. Now, in order to simplify this term, we are going to have to work through some somewhat intricate mathematics. First, we take the boundary to be the zero of some function f , as above. Then, we define $\beta_\perp = \frac{1}{\sqrt{\bar{\nabla}_a f \bar{\nabla}^a f}}$. Then, $r_a = \beta_\perp \bar{\nabla}_a f$, is manifestly normal to the boundary and of unit length. Then, assisted by the identity $\sqrt{\gamma} = \beta_\perp \sqrt{q}$, we can work out:

$$\frac{\partial \mathbf{H}}{\partial \alpha} - \mathcal{C} = - \frac{1}{16\pi G} \int d^2x \bar{\nabla}_a (\sqrt{q} r^a) \tag{5.2.12}$$

$$= - \frac{1}{16\pi G} \int d^2x \bar{\nabla}_a (\sqrt{q} \beta_\perp \bar{\nabla}^a f), \tag{5.2.13}$$

$$= -\frac{1}{16\pi G} \int d^2x \bar{\nabla}_a (\sqrt{\gamma} \bar{\nabla}^a f), \quad (5.2.14)$$

$$= -\frac{1}{16\pi G} \int d^2x (\bar{\nabla}^a f \bar{\nabla}_a \sqrt{\gamma} + \sqrt{\gamma} \bar{\nabla}_a \bar{\nabla}^a f), \quad (5.2.15)$$

$$= -\frac{1}{16\pi G} \int d^2x \left[\frac{1}{\beta_\perp} r^a \bar{\nabla}_a \sqrt{\gamma} + \sqrt{\gamma} \gamma^{ab} (\partial_a \partial_b f - \bar{\Gamma}_{ab}^c \bar{\nabla}_c f) \right], \quad (5.2.16)$$

$$= -\frac{1}{16\pi G} \int d^2x \left(\frac{1}{\beta_\perp} \sqrt{\gamma} r^a \bar{\Gamma}_{ab}^b + \beta_\perp \sqrt{q} \gamma^{ab} (\partial_a \partial_b f - \bar{\Gamma}_{ab}^c \frac{r_c}{\beta_\perp}) \right), \quad (5.2.17)$$

$$= \frac{1}{16\pi G} \int d^2x \sqrt{q} (-r^c \bar{\Gamma}_{ca}^a + \gamma^{ab} r_c \bar{\Gamma}_{ab}^c - \beta_\perp \gamma^{ab} \partial_a \partial_b f), \quad (5.2.18)$$

$$= \frac{1}{16\pi G} \int d^2x \sqrt{q} \left(\frac{1}{2} \gamma^{ab} r^c [(\partial_b \gamma_{ac} + \partial_a \gamma_{cb} - \partial_c \gamma_{ab}) - (\partial_b \gamma_{ac} + \partial_c \gamma_{ab} - \partial_a \gamma_{bc})] \right), \quad (5.2.19)$$

We discarded the term $\gamma^{ab} \partial_a \partial_b f$ in (5.2.18) by requiring that we define the surface on the boundary by a function for which this term vanishes there. This can always be done, since we only require that f vanish on the boundary in order to define f . Therefore, we can always multiply f by another function, and then solve the appropriate Laplace problem in order to find a new f which has a zero on the boundary, but which also satisfies $\gamma^{ab} \partial_a \partial_b f = 0$. This then shows us that the variation of the Hamiltonian with respect to α leaves us with only the term (5.2.19), which is known as the ADM Mass. For asymptotically Cartesian coordinate systems, it is relatively easy to show that this term will very easily yield the mass parameter for simple examples such as the Schwarzschild, Kerr,

and Vaidya metrics.

However, it is very quickly evident that this term is in no way coordinate independent. In particular, if one were to simply write down the metric for Euclidean space in spherical coordinates, and naively compute the “mass”, one would find the following, starting by computing the 3-metric:

$$\gamma_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}.$$

We can define $f = r - r_0$, which makes our surface one of constant r . This then yields $\beta_\perp = 1$, $r^a = (1, 0, 0)$, and $\sqrt{q} = r^2 \sin \theta$. Putting all of this data into (5.2.19), we get the result:

$$\frac{1}{16\pi G} \int d^2x \sqrt{q} (\gamma^{ab} r^c [\partial_a \gamma_{cb} - \partial_c \gamma_{ab}]) = \frac{1}{16\pi G} \int d^2x \left(r^2 \sin \theta \left[-\frac{4}{r} \right] \right) = -\frac{r_0}{G}. \quad (5.2.20)$$

To say the least, this is a silly expression for the amount of mass or energy contained within a sphere of radius r in the Euclidean space! In particular, as we take the limit $r \rightarrow \infty$, it should be clear that this expression diverges. Euclidean space, however, is empty, so we would expect that the value we derive would be zero. Furthermore, if we were to write the Euclidean metric in Cartesian coordinates, it should be manifest that (5.2.19) is equal to zero. We therefore see that this problem is, in fact, a problem with our coordinate system. Older references such as [10] and [13], and [11] will deal with this

apparent contradiction by demanding that formula (5.2.19) is only applicable for asymptotically flat metrics in which the coordinate system asymptotically approaches the Minkowski metric at an appropriate pace as $r \rightarrow \infty$. Following work by Hawking [14], newer references (see [15]) account for this oddity by including a counterterm in the action equal to $+\frac{r_0}{G}$ in the case above, or, more generally, equal to negative one times the ADM Mass of an empty space in the appropriate coordinate system. This fixes this problem by making the formula for the ADM mass coordinate invariant, but it adds the problem of requiring that a fiducial flat metric be introduced into the formalism in order to define the counterterm. Below, an alternative technique to this counterterm will be introduced. It will still retain the problem of requiring a fiducial flat metric, but it will have the advantage of being more compactly defined, and will also have the advantage of being manifestly coordinate invariant.

So, begin by defining the fiducially flat metric $\tilde{\gamma}_{ab}$. It should be defined in such a way that its coordinates are adapted to the spacetime in question—at the minimum, it should have the same coordinate transformation tensor as the true metric does. For example, if you are dealing with a Kerr solution to Einstein’s equation, the fiducially flat metric can be defined by setting the mass parameter of the solution equal to zero. Then, the idea is to treat the true metric as if it were a tensor field living in the flat metric space. Then, we can define a torsion-free derivative operator \mathcal{D}_a that is compatible with the fiducial flat metric (i.e., so that $\mathcal{D}_a \tilde{\gamma}_{bc} = 0$). Once this is done, we use the same function f to define the surface at the boundary of our space, only now,

we use $\tilde{\gamma}_{ab}$ to define a $\tilde{\beta}_\perp$ and \tilde{r}^a . We then do the integral in (5.2.19) over the fiducially flat space, replacing all of the ∂_a terms with \mathcal{D}_a terms. The final result is:

$$M_{ADM,new} = \frac{1}{16\pi G} \int \sqrt{\tilde{q}} d^2x \tilde{\gamma}^{ab} \tilde{r}^c (\mathcal{D}_a \gamma_{bc} - \mathcal{D}_c \gamma_{ab}) . \quad (5.2.21)$$

This expression has several advantages. In the case of asymptotically flat coordinate systems, all of the $\tilde{\Gamma}_{ab}{}^c$'s are all zero, the metric is equal to $\delta_{ab} + O(\frac{1}{r})$ at the surface at infinity, and therefore, it should be clear that (5.2.21) gives the same result as (5.2.19). It should also be clear, moreover, that the integrand of expression (5.2.21) transforms as a scalar under coordinate changes that affect the fiducially flat metric, provided, as stipulated when we defined $\tilde{\gamma}_{ab}$ above, that both metrics have the same coordinate transformation tensor. Therefore, it should be clear that (5.2.21) makes sense if the bulk metric is defined in *any* coordinate system, so long as that coordinate system covers an appropriate two-surface at the spacetime's spacelike boundary.

Furthermore, as a concrete example, consider the Kerr spacetime in the spheroidal coordinates given in Appendix A. Here, the 3-metric is directly obtainable from the 4-metric, and we can find the fiducial metric simply by setting $M = 0$ ⁴:

⁴As always in this work, defining $A = r^2 + a^2$ and $B = r^2 + a^2 \cos^2 \theta$, and using the spheroidal coordinates defined by taking $x = \sqrt{A} \sin \theta \cos \psi$, $y = \sqrt{A} \sin \theta \sin \psi$, and $z = r \cos \psi$, followed by a second coordinate transformation $d\phi = d\psi - \frac{a}{A} dr$. Coordinates are then rendered in the form (r, ϕ, θ) . This form for the metric tremendously simplifies the expression for the metric when compared to the original form, which is a dense 4x4 matrix

$$\begin{aligned}
\gamma_{ab} &= \begin{pmatrix} (1 + \frac{2Mr}{B}) & -(1 + \frac{2Mr}{B})a \sin^2 \theta & 0 \\ -(1 + \frac{2Mr}{B})a \sin^2 \theta & \left(\frac{AB + 2Mra^2 \sin^2 \theta}{B} \right) \sin^2 \theta & 0 \\ 0 & 0 & B \end{pmatrix} \\
\tilde{\gamma}_{ab} &= \begin{pmatrix} 1 & -a \sin^2 \theta & 0 \\ -a \sin^2 \theta & A \sin^2 \theta & 0 \\ 0 & 0 & B \end{pmatrix}.
\end{aligned} \tag{5.2.22}$$

Then, it is simple enough to compute the inverses of these metrics, and follow through with the standard prescription given above, and to compute (5.2.21).⁵ Doing this gives us $\tilde{r}_a = (\sqrt{\frac{B}{A}}, 0, 0)$, and then it only requires the basic definitions of the covariant derivative to compute (dropping the factor of G now, since the metric is written in geometricized units):

$$\begin{aligned}
M_{ADM,new} &= , \\
&= \frac{1}{16\pi} \int d^2x \left(\frac{M \sin \theta [16r^4 + 4a^2 r^2 (\cos(2\theta) + 3) + a^4 (\cos(4\theta) - 1)]}{4 (r^2 + a^2 \cos^2 \theta)^2} \right).
\end{aligned} \tag{5.2.23}$$

The ϕ integration, of course, is easy. All that it does is cancel a factor of 2π in the numerator against the factor of 16π in the denominator. The θ

in both its covariant and contravariant forms.

⁵Of course, we are free to compute (5.2.19), too. Since, however, this metric reduces to a spherical expression of the Schwarzschild Metric for $a = 0$, we would find that this would be beset with the exact same singularity discussed above

integration can be computed with a package such as Mathematica, yielding, amazingly, simply M , independently of which radius is chosen for the integration. This indicates that (5.2.21) has a flavour much more similar to that of Gauss's Law than the expression that you would get from (5.2.19), which would only give you the correct value of M if you were to do the integral on the sphere infinity with the metric expressed in the original Kerr coordinates⁶, or another asymptotically Cartesian form. This technique, however, is both manifestly coordinate invariant and works at an arbitrary radius⁷ without adding explicit counterterms to the action. Furthermore, it automatically yields a zero result for an empty space, simply by the construction of \mathcal{D}_a . The only disadvantage is that it required a massaging of terms after taking a variation, rather than explicitly coming out of a variational principle, in the manner that the terms derived in [11] and [15] were. It also retains the problem, inherent to all of these approaches, of defining some sort of flat metric with which to compare the true metric, whether it is done implicitly by choice of coordinates in the style of the original ADM Paper, or more explicitly, as shown here or in the counterterm approach.

We can apply a similar procedure to the shift vector β_a . Taking only the pure gravity terms in the ADM Hamiltonian involving β_a , we have:

⁶see footnote number (4).

⁷while this slicing is invariant under a 3-dimensional coordinate transformation, and always yields the ADM mass at conformal infinity, at other radii, it is not *slicing* independent—this alternate ADM mass, when computed in a Boyer-Lindquist slicing, gives simply M at infinity, but will monotonically increase to infinity as one approaches the horizon

$$H_{involving\ \beta} = \frac{1}{8\pi G} \left(- \int d^3x \sqrt{\gamma} \beta_b \bar{\nabla}_a \frac{1}{\sqrt{\gamma}} \Pi^{ab} + \oint d^2x \sqrt{q} \frac{1}{\sqrt{\gamma}} r_a \beta_b \Pi^{ab} \right). \quad (5.2.24)$$

Varying this portion of the action with respect to β_a , and remembering that $\Pi^{ab} = \sqrt{\gamma} (K^{ab} - \gamma^{ab} K)$ gives us:

$$\frac{\delta H}{\delta \beta_a} = -\frac{1}{8\pi G} \left[\int d^3x \sqrt{\gamma} \bar{\nabla}_a (K^{ab} - \gamma^{ab} K) - \oint d^2x \sqrt{q} r_a (K^{ab} - \gamma^{ab} K) \right]. \quad (5.2.25)$$

The first term above is easily recognized as the 3-dimensional integral of the Momentum constraint, which must therefore vanish. The second term above, however, does not, in general vanish. In particular, it can be shown to be equal to $M v^i$ for boosted black holes, and equal to $M a \hat{\phi}^i$ for Kerr black holes. Therefore, this term cannot be expected to vanish for a general solution to Einstein's equations. Just as in the Maxwell case above, and for the charged case, however, this can be treated simply by adding a boundary counterterm $\oint d^2x \sqrt{q} (-P^a \beta_a)$ to the Hamiltonian. Then, after the bulk equations of motion have been solved, we can then solve for the appropriate value of P^a such that it cancels the second term in the variation given in equation (5.2.25), thereby giving an overall variation of the Hamiltonian that is equal to zero. Note that each of these counterterms are, in fact, charges corresponding to conserved quantities arising from symmetries in the Hamiltonian/Lagrangian. The Lapse function is associated with time reparameterization invariance of

General Relativity, and therefore, corresponds to an asymptotic energy. Meanwhile, the shift is associated with the choice of 3-dimensional coordinates on the 3+1 fibers. Therefore, the shift vector encodes information about asymptotic translations and rotations. Therefore, the charge, P^i associated with it can be associated with linear and angular momentum.

Chapter 6

Null Geometry

In the following chapters, we are going to be investigating several properties of null submanifolds of general spacetimes. Null spaces have quite a few special properties that are somewhat nonintuitive if one is used to dealing with nondegenerate vector spaces. In particular, there is no natural connection between the intrinsic vector space tangent to a null manifold and the equivalent covector space. Consequently, one must take care in constructing the induced geometry from an enveloping spacetime. For this reason, explicit examples are worked out in Appendix [A](#)

First, remember the techniques derived for projecting onto a surface derived from section [2](#), summarized here for convenience:

1. Find a function f such that $f = \text{constant}$ on the surface in question
2. choose f as a coordinate in the spacetime in question
3. Eliminate the df terms from all relevant forms, and in particular, the

lowered version of the metric tensor. Call this new metric tensor the **induced metric** on the surface

4. Take the inverse of the induced metric, and use this as the raising operator on the surface
5. If there are vectors that need to be projected, you can use one of the following (equivalent) techniques:
 - (a) If the vector already satisfies $v^a df_a = 0$, then the appropriate zero component can simply be dropped, and the vector can just be considered to be a vector living in the fiber. Otherwise:
 - (b) lower the vector using the 4-metric, use the above procedure to project, and then raise using the inverse three-metric
 - (c) define $n_a \equiv \frac{\nabla_a f}{\sqrt{|\nabla_a f \nabla^a f|}}$. Raise n_a using the four-metric. Then project vector indices using the operator $\gamma_a{}^b \equiv \delta_a{}^b - \xi n_a n^b$, as $v^a \gamma_a{}^b$ ¹ will, by construction, have zero components along the appropriate direction, which can then simply be dropped.
6. If it is necessary to project back into the 4-dimensional space, 3-dimensional vectors can simply be included with the appropriate zeros added
7. If you wish to project one-forms back into the 4-dimensional space, however, it is necessary to operate on them using the $\gamma_a{}^b$ operator

¹Recall that ξ is the sign of $n_a n^a$

Now, we are about to need to do the above for the case of a null submanifold of a 4-dimensional space. It might seem trivial to just follow the above steps that supposedly would allow us to seamlessly go back and forth from the 4-space to the 3-space. But on closer glance, step (iv) requires that we be able to invert the induced 3-metric. If the 3-space is null, then the 3-metric necessarily will have an eigenvector with zero eigenvalue, and therefore, will not be invertible. Also, in the null case, the vector n_a defined in (v)(b) will be singular, as $\nabla_a f \nabla^a f = 0$, which means that the projection/inclusion operator $\gamma_a{}^b$ cannot be defined. Therefore, in the null case, some extra care must be taken in the above procedures when working out the appropriate induced geometry.

So, first, we start by noting that, until the fourth step, there was nothing problematic at all. In particular, we still consider our surface to be a solution set for some equation. In particular, for a Kerr spacetime in spheroidal Kerr coordinates, we consider the horizon to be the solution to the equation $r^a + a^2 - 2Mr = 0$. This makes the first few steps easy enough to follow. In particular, this means that we can map forms in the 4-dimensional spacetime to forms on the null submanifold using the same procedure (i.e., by dropping the r component of the form in the Kerr spacetime). Note, however, that this technique will already create some anomalies for null submanifolds. In particular, it causes the lowered version of the outgoing null normal to the Kerr horizon to be projected to the zero vector on the horizon. This might lead to some mystery regarding what the three covectors that span the dual

tangent space to the Kerr horizon are, but before this mystery is settled, let us soldier along.

First, let us denote the 3-metric by q_{ab} ². If the form that we omitted in order to define q_{ab} was null according to the 4-metric, then it is the case that q_{ab} can be shown to have zero determinant. This then means that there is a three vector that is mapped to the zero form by q_{ab} . To ascertain the nature of this vector, note that, by the fact that df_a is null, that $df_a g^{ab}$ will have no component along the ∂_f direction. We therefore can infer that the vector that is mapped to the zero covector by q_{ab} is, in fact, $g^{ab} df_b$, times a function, which we will henceforward refer to as the outgoing null normal ℓ^a .

Now, let us return to looking at the basis of the covector space. If we use q_{ab} to evaluate the norms of the vectors in the tangent plane to the null surface, we can see, relatively quickly, that the tangent plane to the null surface is spanned by ℓ^a and two other spacelike vectors. Therefore, we would expect to have the covector space spanned by three vectors that, when acting upon unit vectors in the tangent space, will give you Kroneker delta functions as output. Since the raised versions are already spacelike under the action of q_{ab} , for the spacelike vectors, we can simply raise and lower indices using the metric of the horizon just like any other vector. $q_{ab} \ell^b = 0$, however, so we cannot do this. And we already established that the 4-dimensional covector

²There is some trickiness here—it is typical practice to *also* define the spacelike metric of the 2-dimensional sections of the horizon defined by constant null parameter as q_{ab} . We follow normal convention here, which is at least aided by the fact that q_{ab} is degenerate with a zero eigenvalue being precisely the difference between the 2-space and the 3-space. Regardless, the reader is advised to be careful when interpreting something written down as q_{ab} .

ℓ_a has a vanishing pullback onto the null surface. However, if we specify two spacelike vectors and ℓ_a , then there is a unique null covector k_a such that it is orthogonal to the two spacelike vectors and $g^{ab}\ell_a k_b = -1$. The pullback of this vector onto the horizon (denoted $\underset{\leftarrow}{k_a}$) will be the third one-form spanning the covector space. Then, we will require that $q^{ab}\underset{\leftarrow}{k_a} = 0$, which guarantees that q^{ab} is degenerate, as well as uniquely specifying q^{ab} . Finally, we conclude this section by pointing out that we now can define the projection operator of vectors onto the null surface as: $P_a^b \equiv -k_a \ell^b + q^{ac} q_{cb}$, which manifestly has no effect upon the two spacelike vectors normal to k_a and ℓ_a , and has no effect on ℓ^a , but maps k^a to zero.

Now, with both q^{ab} and q_{ab} uniquely specified, we can define the intrinsic connection $\tilde{\nabla}_a$ to the null surface according to (for v^b tangent to the null surface) $\tilde{\nabla}_a v^b \equiv \partial_a v^b + \tilde{\Gamma}_{ab}^c$, where $\tilde{\Gamma}_{ab}^c \equiv \frac{1}{2} q^{cm} (q_{am,b} + q_{bm,a} - q_{ab,m})$. This concludes our section on the embeddings of null submanifolds. We now progress to analyze Ashtekar's Isolated and Dynamical Horizon formalism. For the sake of clarity, in the appendix, we work out this decomposition for two null surfaces: a null cone in Minkowski spacetime and the event horizon of a Kerr spacetime.

6.1 Null Expansions

Now that we have defined our two null normals and come up with an unambiguous way to define induced geometries on null surfaces, let us consider a few properties of these null surfaces. One quantity that will repeatedly be of

critical importance in the following proofs will be the notion of the expansion of a null normal. As was shown above, the covector space to the horizon will contain one null one-form, labeled k_a , while the vector space will contain a vector ℓ^a that is not metrically related to k_a in the enveloping spacetime. We will denote k_a as the ingoing null normal, and ℓ_a as the outgoing null normal. Note that ℓ_a is the null generator of the horizon. Now, consider the null expansions of these two vectors, defined by $\theta = q^{ab}\nabla_a\ell_b$ and $\theta_{(k)} = q^{ab}\nabla_ak_b$. These will have a few interpretations in this work, but for now, we will interpret them as the time rate of change of two surfaces along flowlines of the relevant vector. If one doubts this interpretation, it is easy to show that this is in fact valid:

$$\begin{aligned}
\mathcal{L}_\ell A &= \mathcal{L}_\ell \int \sqrt{q} = \int \frac{1}{2\sqrt{q}} \mathcal{L}_\ell q, \\
&= \frac{1}{2} \int \sqrt{q} q^{ab} \mathcal{L}_\ell q_{ab} = \frac{1}{2} \int \sqrt{q} q^{ab} (\ell^c \partial_c q_{ab} + 2q_{ac} \partial_b \ell^c), \\
&= \frac{1}{2} \int \sqrt{q} (q^{ab} \ell^c \partial_c q_{ab} + 2q^{ab} q_{ac} (\nabla_b \ell^c - \Gamma_{bd}{}^c \ell^d)), \\
&= \frac{1}{2} \int \sqrt{q} (2\theta_{(\ell)} + q^{ab} \ell^c \partial_c q_{ab} - q^{ab} q_{ac} \ell^d g^{cm} (g_{bm,d} + g_{dm,b} - g_{bd,m})), \\
&= \frac{1}{2} \int \sqrt{q} (2\theta_{(\ell)} + q^{ab} \ell^c \partial_c q_{ab} - q^{mb} \ell^d (g_{bm,d} + g_{dm,b} - g_{bd,m})), \\
&= \frac{1}{2} \int \sqrt{q} (2\theta_{(\ell)} + q^{ab} \ell^c \partial_c q_{ab} - q^{mb} \ell^d \partial_d (q_{bm} - \ell_b k_m - \ell_m k_b)), \\
&= \int \sqrt{q} (\theta_{(\ell)}). \tag{6.1.1}
\end{aligned}$$

Therefore, if $\theta_{(\ell)}$ is constant on the two-surface, which will typically be the case in this work, then the rate of change of that surface's area as one Lie drags along ℓ^a is simply $\theta_{(\ell)}A$, and if $\theta_{(\ell)} = 0$, then the surfaces have a constant area.

In the dynamical case, this argument will break down, but the reason for that is that ℓ^a will cease to be a tangent vector of the dynamical horizon, which will be spacelike, rather than null.

6.2 Raychaudhuri's equation

In this section, we will derive Raychaudhuri's equation for the case of a null vector. This result will be used frequently throughout this text, so it is useful to show its origin here.

Consider a non-affinely parameterized null vector ℓ^a . Define its affine parameter by $\ell^a \nabla_a \ell^b = \kappa \ell^b$. Furthermore, define a second associated null vector k^a such that $g_{ab} k^a \ell^b = -1$, and raise and lower indices on ℓ^a and k^a freely using the metric tensor and its inverse. Then, the operator $q_{ab} \equiv g_{ab} + \ell_a k_b + k_a \ell_b$ annihilates both ℓ^a and k^a , as can be verified by inspection. We wish to consider q_{ab} to, in a sense, be the metric of a two surface, but note that it is degenerate³, and therefore, has no natural inverse. We therefore must raise and lower indices on q_{ab} using the full metric tensor, as is customarily the case when dealing with objects in a null geometry.

Now, consider the tangent gradients of ℓ_a given by $q_a{}^c q_b{}^d \nabla_c \ell_d$. Obviously, this is a tensor with two indices, and therefore, can be decomposed into an antisymmetric part, a traceless symmetric part, and a trace, as shown below:

³Having, after all, two linearly independent null vectors!

$$q_a{}^c q_b{}^d \nabla_c \ell_d \equiv \frac{1}{2} q_{ab} \theta + \sigma_{ab} + \omega_{ab}. \quad (6.2.1)$$

where the factor of $\frac{1}{2}$ is included so that $q^{ab} \nabla_a \ell_b = \theta$. It will be of great interest to evaluate the evolution of quantities along flowlines of ℓ^a , which will usually be taken to be the null tangent vector of the horizon. Furthermore, the quantity θ will prove to be the critical central element in the definition of Isolated and Dynamical horizons. Therefore, it is natural to want to calculate the quantity $\mathcal{L}_\ell \theta$. So, let us now do so⁴. Note that similar derivations can be found in several sources, including [13] [15]

$$\begin{aligned} \mathcal{L}_\ell \theta &= \mathcal{L}_\ell (q^{ab} \nabla_a \ell_b) = q^{ab} \mathcal{L}_\ell \nabla_a \ell_b + (\nabla_a \ell_b) \mathcal{L}_\ell q^{ab}, \\ &= q^{ab} \ell^c \nabla_c \nabla_a \ell_b + q^{ab} (\nabla_c \ell_b) \nabla_a \ell^c + q^{ab} (\nabla_a \ell_c) \nabla_b \ell^c \\ &\quad + (\nabla_a \ell_b) (\ell^c \nabla_c q^{ab} - q^{cb} \nabla_c \ell^a - q^{ac} \nabla_c \ell^b), \\ &= q^{ab} \ell^c R_{cab}{}^d \ell_d + q^{ab} \ell^c \nabla_a \nabla_c \ell_b + q^{ab} (\nabla_c \ell_b) \nabla_a \ell^c + q^{ab} (\nabla_a \ell_c) \nabla_b \ell^c \\ &\quad + (\nabla_a \ell_b) (\ell^c \nabla_c (\ell^a k^b + \ell^b k^a) - q^{cb} \nabla_c \ell^a - q^{ac} \nabla_c \ell^b), \\ &= -R_{ab} \ell^a \ell^b + q^{ab} \nabla_a (\ell^c \nabla_c \ell_b) - q^{ab} (\nabla_a \ell^c) \nabla_c \ell_b + q^{ab} (\nabla_a \ell^c) \nabla_c \ell_b \\ &\quad + q^{ab} (\nabla_b \ell^c) \nabla_a \ell_c + (\nabla_a \ell_b) \ell^a \ell^c \nabla_c k^b + (\nabla_a \ell_b) k^b \ell^c \nabla_c \ell^a \\ &\quad + (\nabla_a \ell_b) \ell^c k^a \nabla_c \ell^b - q^{cb} (\nabla_a \ell_b) \nabla_c \ell^a - q^{ac} (\nabla_a \ell_b) \nabla_c \ell^b, \\ &= -R_{ab} \ell^a \ell^b + q^{ab} \nabla_a (\kappa \ell_b) + q^{ab} (\nabla_b \ell^c) \nabla_a \ell_c + \kappa \ell_b \ell^c \nabla_c k^b \end{aligned}$$

⁴Note that $g^{ab} R_{acbd} \ell^c \ell^d = q^{ab} R_{acbd} \ell^c \ell^d = R_{ab} \ell^a \ell^b$, since all of the terms by which g^{ab} and q^{ab} differ include factors of ℓ^a , and the antisymmetry of the Riemann tensor will guarantee that these terms vanish.

$$\begin{aligned}
& + \kappa \ell^a k^b \nabla_a \ell_b + \kappa \ell^b k^a \nabla_a \ell_b - q^{ab} (\nabla_c \ell_b) \nabla_a \ell^c - q^{ab} (\nabla_a \ell_c) \nabla_b \ell^c, \\
& = -R_{ab} \ell^a \ell^b + \kappa \theta - \kappa k^b \ell^c \nabla_c \ell_b + \kappa \ell^a k^b \nabla_a \ell_b - q^{ab} (\nabla_c \ell_b) \nabla_a \ell^c, \\
& = -R_{ab} \ell^a \ell^b + \kappa \theta - \left(\frac{1}{2} \theta q^{ba} + \sigma^{ba} + \omega^{ba} \right) \left(\frac{1}{2} \theta q_{ab} + \sigma_{ab} + \omega_{ab} \right) \\
& = -8\pi T_{ab} \ell^a \ell^b + \kappa \theta - \frac{1}{2} \theta^2 - \sigma^{ab} \sigma_{ab} + \omega^{ab} \omega_{ab}. \tag{6.2.2}
\end{aligned}$$

where the Einstein equation was used in the last line in order to convert the Ricci tensor into a stress energy tensor.

Equation (6.2.2) is the null Raychaudhuri equation. It is of critical importance in several applications involving various horizons in relativity. In particular, its use was critical in the singularity proofs of Hawking and Penrose that showed that the existence of a closed trapped surface⁵ along with a reasonable restriction on the matter content of spacetime, you will necessarily have a spacetime singularity.

⁵i.e., one for which, given a closed 2-surface and its two associated null vectors, ℓ^a and k^a , you have $\theta_{(\ell)} \leq 0$ and $\theta_{(k)} < 0$.

Chapter 7

Isolated Horizons

Now, let us consider the Isolated and Dynamical Horizon¹ framework as developed by Ashtekar and others. Previous attempts to define black holes have been fraught with difficulty. The most typical definition of the boundary of the black hole was the event horizon, which is defined as the past development of timelike future infinity. While this definition has a convenient feature of capturing the nature of ‘the point of no return’ that is typically associated with black holes, it also faces the significant problem that it is defined only globally in a spacetime. Therefore, locating a spacetime’s event horizon requires knowledge of the entire future development of that spacetime. In particular, there are simple analytical examples (i.e., the Vaidya metric where $M(v)$ is nonconstant for a compact subdomain of v ²) where event horizons ex-

¹Henceforward, we will use the abbreviation IH to denote Isolated Horizon, and DH to denote Dynamical Horizon.

²As a concrete example, take $M(v) = 0$ for $v < 0$, $M(v) = M_0 v$ for $0 < v < 1$ and $M(v) = M_0$ for $v > 1$, as shown in figure 7.1. There will be regions in the past domain of dependence of the late-time singularity (More succinctly, points inside of the event horizon)

ist in flat regions of spacetime. Therefore, local observers will have no means by which to identify whether or not an event horizon is present. Furthermore, beyond this conceptual ‘epistemological’ objection, there is the much more practical problem that the global definition of the event horizon makes it difficult to locate its intersection with a particular Cauchy slice of a spacetime, and therefore, very difficult to locate numerically (and in fact, for the Vaidya example discussed above, the event horizon *cannot* be located using only quantities defined locally on that particular Cauchy slice, as the points for which $v < 0$ were exactly isomorphic to finitely sized regions of Minkowski spacetime.). However, having at our disposal all of the machinery of isometric embeddings and manifold theory in the DH case, while the behaviour of the horizon can still be, in the abstract, wild and superluminal, we now know something of how to track the motion of the DH across the Cauchy slice.

Therefore, the focus began to be shifted toward finding a definition of the ‘point of no return’ that could be defined locally, or at least quasi-locally. The first attempt to do so was done by Hawking, with his notion of Apparent Horizon, which he defined as the boundary of a ‘trapped region’. This was an improvement over the event horizon as a notion, as it is definable at a moment of time. It still proved problematic, however, as it does require some use of the non-local notion of a ‘trapped region’. In particular, this problem would lead to several cases where the behaviour of the apparent horizon proved to be ‘wild’—apparent horizons would move and evolve discontinuously in certain

that will have completely flat intrinsic and extrinsic geometries. Observers at these times will have no way of knowing that they are inside of an event horizon.

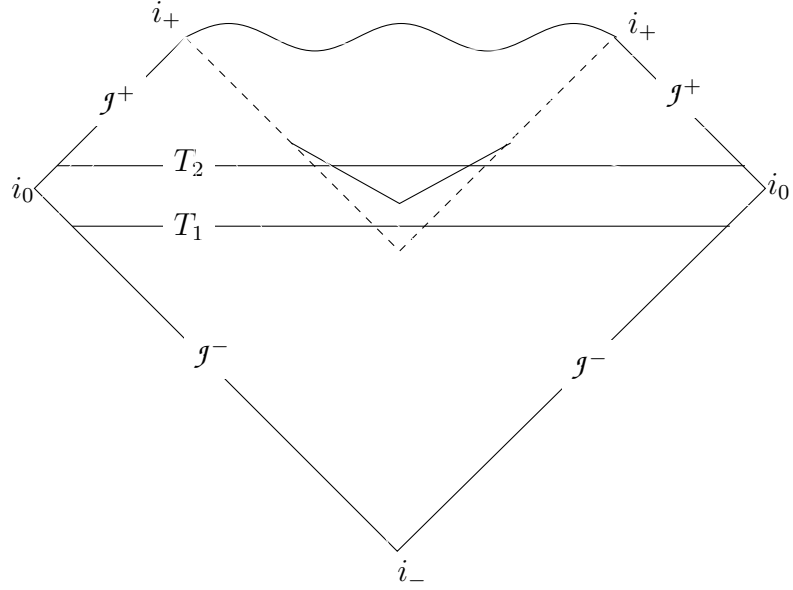


Figure 7.1: A Penrose-Carter Diagram of a Vaidya spacetime with a mass function that is linear for a finite advanced time. The dashed line represents the event horizon, and the solid line in the interior of the diagram represents the dynamical horizon. T_1 represents a spacelike slice which intersects the event horizon, but has zero extrinsic curvature and a Euclidean metric. T_2 represents a spacelike slice that intersects the dynamical horizon. All slices after the dynamical horizon merges with the event horizon are identical to slices of Schwarzschild.

spacetimes for certain slicings. Still, they have proven useful, and have been implemented numerically.

All of these problems, however, have been removed by the closely related notions of Hayward’s trapping horizon, and Ashtekar et al.’s notion of Isolated and Dynamical Horizons. These notions require merely that the notion of non-expansion be confined to a single null or spacelike submanifold of the full spacetime. In particular, this guarantees that the wild behaviour discussed above can be avoided—since the horizon is defined to be a smooth manifold, the behaviour of its slicings is guaranteed to also be smooth. It is therefore considered to be quite compatible with a great number of approaches in Relativity, including numerical ones. Using the machinery developed in the previous chapter, we now work to develop this framework in detail. We start with the much simpler and restrictive notion of Isolated Horizon, and move on to the Dynamical horizon. Finally, this chapter will conclude with a proof of the Laws of black hole dynamics using this formalism, and then with a concrete application to the Vaidya spacetime.

Now, consider a 3-manifold that is a subspace of a Lorentzian 4-manifold. Following Ashtekar and Krishnan, this three manifold will be considered to be a non-expanding horizon if the following three conditions are met:

1. The 3-manifold is null, topologically $\mathbb{R} \times \mathbb{S}^2$ or $\mathbb{R} \times \mathbb{T}^2$ and has the null vector ℓ^a as a tangent vector, and associated (degenerate) metric q_{ab}
2. On the three-manifold, the expansion $\theta_{(\ell)} = q^{ab} \nabla_a \ell_b$ vanishes
3. The 4-metric g_{ab} satisfies Einstein’s equation, and all relevant matter

equations of motion are satisfied, and any matter involved obeys the Dominant Energy Condition.

The third condition just specifies that we are in fact talking about General Relativity, and the Dominant Energy Condition guarantees the vanishing of certain terms in the Raychaudhuri equation, as will be seen below. Meanwhile, the first condition guarantees that you have a static, non-evolving horizon, as translation along the null vector ℓ^a leaves one on the three-manifold. The essential condition, therefore, is the second one, which captures the essence of the Horizon being the outermost ‘point of no return’: The null geodesics that have tangent vector ℓ^a simply transport null observers along a horizon of constant area³, rather than allow them to move spatially, as would typically be the case for a ‘typical’ null hypersurface, like a null-cone in Minkowski spacetime. In particular, this guarantees that, for some function κ , defined on the horizon, $\ell^a \nabla_a \ell^b = \kappa \ell^b$. There will have much more to say about κ further along.

Now, consider the Null Raychaudhuri equation:

$$\ell^a \tilde{\nabla}_a \theta_{(\ell)} = -\frac{1}{2} \theta_{(\ell)}^2 - \sigma_{ab} \sigma^{ab} + \omega_{ab} \omega^{ab} - 8\pi T_{ab} \ell^a \ell^b . \quad (7.0.1)$$

Where $\sigma_{ab} \equiv P_a^m P_b^n \nabla_{(m} \ell_{n)} - \frac{1}{2} q_{ab} \theta_{(\ell)}$, and $\omega_{ab} \equiv P_a^m P_b^n \nabla_{[m} \ell_{n]}$. Since we know that the horizon is defined by the fact that $\theta_{(\ell)} = 0$, and that it has ℓ^a as a null tangent vector, we know that the left hand side of this equation vanishes on the horizon. Furthermore, we know that ω_{ab} vanishes, since the

³One might wonder about this statement regarding area. See (6.1)

pullback of ℓ_a onto the horizon also vanishes. Thus, the above equation reduces to:

$$0 = \sigma_{ab}\sigma^{ab} + 8\pi T_{ab}\ell^a\ell^b. \quad (7.0.2)$$

Since both of these terms are manifestly positive (the horizon metric is non-negative-definite, and the null energy condition is satisfied), this means that they both must independently vanish. Now, since the decomposition of $P_a{}^m P_b{}^n \nabla_a \ell_b \equiv \tilde{\nabla}_a \ell_b$ is:

$$\tilde{\nabla}_a \ell_b = \frac{1}{2} q_{ab} \theta_{(\ell)} + \sigma_{ab} + \omega_{ab}, \quad (7.0.3)$$

$$= \sigma_{ab}, \quad (7.0.4)$$

and since $\ell_b \nabla_a \ell^b = 0$ along with $\omega_{ab} = \sigma_{ab} = \theta_{(\ell)} = 0$, it must be the case that all contractions of $q^{ma} q_{nb} \nabla_a \ell^b = \tilde{\nabla}_m \ell^n = 0$. Therefore, we must have, on the horizon only,

$$\nabla_a \ell^b = \omega_a \ell^b + \tilde{\omega}_a k^b.$$

Since ℓ^a has a fixed norm, however, we know that $\ell_a \nabla_b \ell^a = 0$, since it is equal to minus itself by the product rule. Therefore, contracting the above expression for $\nabla_a \ell^b$ onto ℓ_b gives us the result that $\tilde{\omega}_a = 0$. Therefore, on the horizon, we have the condition:

$$\nabla_a \ell^b = \omega_a \ell^b \dots \quad (7.0.5)$$

Where ω_a is called the Hájiček one-form, and should be taken as a three component object intrinsic to the horizon. Also, it should be clear that $\kappa = \ell^a \omega_a$. In computations, it is often simpler to take advantage of the fact that $\ell^a k_a = -1$, and work with $\omega_a = -k_b \nabla_a \ell^b$ since, among other things, the meaning of the pullback operator is clear, as one can compute the action of the derivative operator, perform the index contraction and **then** perform the pullback operation.

Now, note that all of the above is completely invariant if we replace (ℓ_a, k_a) with (L_a, K_a) using the rescaling transformation:

$$L_a = f \ell_a \quad K_a = \frac{1}{f} k_a. \quad (7.0.6)$$

Since all results above were merely dependent on the inner product $\ell_a k^a$ being equal to minus one, and each of them being a null vector, and both of these are true of you rescale ℓ_a by an arbitrary function f . Note, however, that when you apply this rescaling transformation, you induce a change in ω_a :

$$K_b \nabla_a L^b = \frac{1}{f} k_b \nabla_a (f \ell^b), \quad (7.0.7)$$

$$= k_b \nabla_a \ell^b - \frac{1}{f} \nabla_a f. \quad (7.0.8)$$

Which, after pulling back onto the horizon, gives us the result

$$\omega_a \rightarrow \omega_a + \nabla_a \ln(f). \quad (7.0.9)$$

And, since $\kappa = \ell^a \omega_a$, we also get $\kappa \rightarrow f\kappa + \mathcal{L}_\ell f$. This rescaling freedom gives us the ability to, without loss of generality, impose more stringent conditions upon our horizon. In particular, if we choose an appropriate function, we can impose the condition that ω_a not evolve along flowlines of ℓ^a . More formally, if we make this choice, we can declare that the horizon be a “Weakly Isolated Horizon” (WIH) by saying that it satisfies:

$$0 = [\mathcal{L}_\ell, \nabla_a] \ell^b = \mathcal{L}_\ell(\omega_a \ell^b) = \ell^b \mathcal{L}_\ell \omega_a. \quad (7.0.10)$$

Where we twice used the condition that $\mathcal{L}_\ell \ell^a = 0$. Now, the role of the rescaling freedom becomes clear. If ω_a does not satisfy (7.0.10), then we simply choose to rescale ℓ_a and k_a by some f according to (7.0.6), and solve the differential equation for f in such a way that (7.0.10) is satisfied. This will then fix the function f up to a positive constant (since ℓ^a needs to remain future-pointing and nonzero). Once this choice is made, it will impose the condition that κ be constant on the Horizon (though it does not fix the nonzero value of κ). This result is called the Zeroth Law of Black Hole Dynamics.

7.1 The First Law of Black Hole Dynamics

7.1.1 A Simple Proof of the First Law

We can use the isolated and dynamical horizon rules to derive the first law of black hole dynamics. The proof will be somewhat laborious, however, and rather than directly going into it, we will instead derive the first law quickly

here, and then generate the full proof. The below proof will have the advantage of being direct and easy. It will have the disadvantage of being very dependent on the details of the Kerr solution. This simpler proof is outlined in [15]

Now, consider the Kerr spacetime, with metric as given in Appendix A. We define the horizon to be the surface at which $A - 2Mr = r^2 + a^2 - 2Mr = 0$. Each closed 2-dimensional section will also have a constant t coordinate. Using the decomposition formalism built up in Chapter 6, it is apparent that the 2-dimensional metric of this surface is given by:

$$q_{AB} = \begin{pmatrix} \frac{A_+^2 \sin^2(\theta)}{B_+} & 0 \\ 0 & B_+ \end{pmatrix}. \quad (7.1.1)$$

Where the coordinates are (ϕ, θ) and A_+ and B_+ denote that the functions A and B as defined in Appendix A take the values of r required by the equation $r^2 - 2Mr - a^2 = 0$. It is now easy to compute the area of the 2-dimensional surface:

$$\begin{aligned} A &= \int d^2x \sqrt{q} = \int_0^\pi d\theta \int_0^{2\pi} d\phi A_+ \sin(\theta) = 4\pi A_+, \\ &= 4\pi (r_+^2 + a^2) = 4\pi \left(M^2 + 2M\sqrt{M^2 - a^2} + M^2 - a^2 + a^2 \right), \\ &= 4\pi \left(2M^2 + 2M\sqrt{M^2 - a^2} \right). \end{aligned} \quad (7.1.2)$$

Meanwhile, it is easy to calculate the angular momentum in the Kerr spacetime. Remembering the boundary formulation given in (5.2.2), we see

that it is necessary to add a term $\oint d^2x P^i \beta_i$ to the ADM Hamiltonian. Where P^a is given by the solution valued integral:

$$P^i = \oint d^2x \bar{r}_j (K^{ij} - \gamma^{ij} K) . \quad (7.1.3)$$

In order to find the angular momentum, we look for the ϕ component of this vector. Due to the vagaries of vector valued integrals, we must multiply by a unit vector in the ϕ direction under the integral. After multiplying by this vector, and then taking the limit to infinity, we get:

$$J = P^\phi = M a \rightarrow a = \frac{J}{M} . \quad (7.1.4)$$

Now, we insert this relationship into the expression (7.1.2) to eliminate the dependence on a and replace it with dependence on J . The result is:

$$\frac{1}{8\pi} A = M^2 + \sqrt{M^4 - J^2} . \quad (7.1.5)$$

Now, we take the variation of this equation, treating M, J , as independent variables with independent variations:

$$\begin{aligned} \frac{1}{8\pi} \delta A &= 2M \delta M + \frac{1}{2} \left(\frac{4M^3 \delta M - 2J \delta J}{\sqrt{M^4 - J^2}} \right) , \\ \frac{1}{8\pi} \delta A + \frac{J \delta J}{\sqrt{M^4 - J^2}} &= \delta M \left(\frac{2M \sqrt{M^4 - J^2} + 2M^3}{\sqrt{M^4 - J^2}} \right) , \\ \delta M &= \frac{1}{8\pi} \delta A \left(\frac{\sqrt{M^4 - J^2}}{2M \sqrt{M^4 - J^2} + 2M^3} \right) , \end{aligned}$$

$$\begin{aligned}
& + \delta J \left(\frac{J}{2M\sqrt{M^4 - J^2} + 2M^3} \right) \\
& = \frac{1}{8\pi} \delta A \left(\frac{\sqrt{M^2 - a^2}}{2M^2 + 2M\sqrt{M^2 - a^2}} \right) \\
& \quad + \delta J \left(\frac{a}{2M^2 + 2M\sqrt{M^2 - a^2}} \right), \\
& = \frac{1}{8\pi} \kappa \delta A + \Omega \delta J.
\end{aligned} \tag{7.1.6}$$

Where the multiplier κ of $\frac{1}{8\pi}\delta A$ is the surface gravity of the horizon, defined from the two null normals ℓ^a and k^a to the horizon by $\kappa = -\ell^a k^b \nabla_a \ell_b$, while Ω is the rotation parameter to the horizon, most directly defined as the value that $\frac{g^{r\phi}}{g^{\phi\phi}}$ takes on the horizon⁴. The generalization to the Kerr-Reissner-Nordstrom case is straightforward, and in the end will only involve the addition of a $\Phi\delta Q$ term, where Q is the charge parameter of the black hole, and Φ is the electrostatic potential at the horizon.

Several things should be noted about this derivation. First, it was heavily dependent on the Kerr solution. We had to explicitly calculate the area of the Kerr horizon, and then take its variation. Therefore, this can really only be interpreted as a statement about the phase space of Kerr (Reissner-Nordstrom) spacetimes, and perturbations thereof, and not a general statement about black holes. Furthermore, it required some odd mixtures of terms—the mass, charge and angular momentum, in standard formulations, are properly defined by integrals at infinity. We performed one of these above in order to obtain the expression $J = aM$. On the other hand, Ω , κ , Φ , and A are proper-

⁴And, thus, in a naïve sense, the “rate of rotation” of the space around the horizon.

ties inherently intrinsic to the horizon itself. They would, in principle, be computable even in spacetimes without boundary so long as those spacetimes contained closed trapped surfaces. It seems weird to have a single equation that multiplies these objects together haphazardly without much regard to which surface they belong. Older, more general formulations of the first law also have found themselves beset by this problem. And it is the task of the Isolated and Dynamical Horizon formulation to remove these problems by generating a formulation of the First Law that is both not dependent on the details of a particular solution and also in terms of quantities wholly defined on the horizon. In order to do this, however, we must continue further into the Isolated/Dynamical Horizon rabbit hole in order to build up some tools.

Chapter 8

Decomposition of quantities on the horizon into ADM quantities

Another key use of the IH/DH formalism is the ability that it gives you to decompose 3+1 quantities into quantities well defined on the horizon. Now, consider the case of either an isolated or dynamical horizon. In general, a spacelike section of a 3+1 split will intersect an isolated or dynamical horizon in 2-dimensional spacelike slices. Each of these slices will therefore have a spacelike normal \bar{s}^a lying in the 3+1 slice, and will also be perpendicular to the timelike normal n^a used to perform the 3+1 slicing. Therefore, we can make the usual choice $\ell^a = \alpha(n^a + \bar{s}^a)$ to decompose the outgoing null vector to the horizon, where the proportionality to α is chosen so that ℓ^a mirrors the time vector t^a as closely as possible. We will first develop some generic formalism that works for all zero expansion surfaces. Then, we will work out some quantities specialized to the Isolated and Dynamical cases.

8.1 Doubly decomposing ADM quantities on the horizon

It should be clear that the project described above, where one defines an ADM splitting, and then, within the context of that splitting, divides the 3-space into the tangents and normals to an intersection of the horizon with the particular 3+1 slice, is exactly the sort of the double foliation discussed in Appendix D. The formalism developed above can now be used to derive powerful relationships between the intrinsic horizon quantities and the intrinsic 3+1 quantities. In order to accomplish this, we are going to need to work out a few identities. First, recall from equation (3.1.16) that

$$\nabla_a n_b = -n_a \bar{\nabla}_b \ln(\alpha) - K_{ab}$$

It is our goal to decompose the quantity $\nabla_a \bar{s}_b$ in the same way. In order to do this, we write down the quantity, and then expand the 4-metric in terms of projection operators and normal operators:

$$\begin{aligned} \nabla_a \bar{s}_b &= g_a^c g_b^d \nabla_c \bar{s}_d, \\ &= (\gamma_a^c - n_a n^c) (\gamma_b^d - n_b n^d) \nabla_c \bar{s}_d, \\ &= \gamma_a^c \gamma_b^d \nabla_c \bar{s}_d - \gamma_a^c n_b n^d \nabla_c \bar{s}_d - n_a n^c \gamma_b^d \nabla_c \bar{s}_d + n_a n_b n^c n^d \nabla_c \bar{s}_d, \\ &= \bar{\nabla}_a \bar{s}_b + \gamma_a^c n_b \bar{s}_d \nabla_c n^d - n_a n_b n^c \bar{s}_d \nabla_c n^d - n_a n^c \gamma_b^d \nabla_c \bar{s}_d, \\ &= \bar{\nabla}_a \bar{s}_b - n_b \bar{s}_d K_a^d - n_a n_b \bar{s}_d \bar{\nabla}^d \ln(\alpha) - n_a \gamma_b^d n^c \nabla_c \bar{s}_d. \end{aligned} \tag{8.1.1}$$

So, we have decomposed the first three terms relatively easily in terms of 3+1 quantities. Furthermore, since the argument used in deriving equation (3.1.16) made no assumptions about the dimension of the space involved nor the signature of the normal vector, we can further decompose the first term into $-H_{ab} - \bar{s}_a \hat{\nabla}_b \ln(\bar{B})$, where H_{ab} is the extrinsic curvature of the horizon section in the 3+1 slice, and \bar{B} is the generalized Lapse function of the 2+1 foliation in the neighborhood of the horizon¹. The last term above, however, requires a more intricate analysis. First, we need to calculate the inner product of the unbarred radial vector with the normal vector² in terms of 3+1 variables:

$$\begin{aligned}
n^a s_a &= g^{ab} n_a s_b = \alpha B g^{ab} (\nabla_a \tau) (\nabla_b R) , \\
&= \alpha B (g^{ab} \nabla_a \tau) \nabla_b R , \\
&= \alpha B \left(-\frac{1}{\alpha^2} n^b + \frac{1}{\alpha^2} \beta^b \right) \left(\frac{1}{\bar{B}} \bar{s}_b \right) , \\
&= \left(\frac{B}{\alpha \bar{B}} \right) \beta^a \bar{s}_a .
\end{aligned} \tag{8.1.2}$$

Now, we compute the value of $-\gamma_b^d n^c \nabla_c \bar{s}_d$. Before moving on, note that the fixed norm of \bar{s}^a means that the contraction of this term onto \bar{s}^b vanishes. Therefore, we can replace the 3-metric appearing in this term with the 2-metric. Now, we compute this term, taking advantage of the lack of

¹Less formally, $\bar{B} = \frac{1}{\sqrt{\gamma_{RR}}}$, where $R = \text{constant}$ determines the location of the horizon, and is chosen as a coordinate on the 3+1 slice

²See the first section of this chapter for more clarity here. In essence, the unbarred radial vector is $s_a = B \nabla_a R$, where B is chosen in such a way as to make s_a a unit normal relative to g^{ab} while $\bar{s}_a = \bar{B} \nabla_a R$ is a unit normal relative to γ^{ab} .

torsion to interchange the order of derivatives of a scalar:

$$\begin{aligned}
-\gamma_b{}^d n^c \nabla_c \bar{s}_d &= -q_{bd} n^c \nabla_c (\bar{B} \gamma^{de} \nabla_e R) , \\
&= -q_{bd} n^c \bar{B} \gamma^{de} \nabla_c \nabla_e R - q_{bd} n^c \bar{B} (\nabla_e R) \nabla_c \gamma^{de} \\
&\quad - q_{bd} n^c \gamma^{de} (\nabla_e R) \nabla_c \bar{B} , \\
&= -q_b{}^e n^c \bar{B} \nabla_c \nabla_e R - q_{bd} n^c \bar{B} (\nabla_e R) (n^d \nabla_c n^e + n^e \nabla_c n^d) \\
&\quad - q_{bd} n^c \bar{s}^d \nabla_c \ln(\bar{B}) , \\
&= -q_b{}^e n^c \bar{B} \nabla_e \nabla_c R - \bar{B} q_{bd} n^c n^e (\nabla_e R) \nabla_c n^d , \\
&= -\bar{B} q_b{}^e \nabla_e (n^c \nabla_c R) + \bar{B} q_b{}^e (\nabla_e n^c) \nabla_c R - \bar{B} q_{bd} (n^e \nabla_e R) n^c \nabla_c n^d , \\
&= -\bar{B} q_b{}^e \nabla_e \left(\frac{1}{\bar{B}} n^c s_c \right) - q_b{}^c \bar{s}^d K_{cd} - \frac{\bar{B}}{B} q_{bd} n^c s_c \bar{\nabla}^d \ln(\alpha) , \\
&= -\bar{B} q_b{}^c \nabla_c \left(\frac{1}{\alpha \bar{B}} \beta^a \bar{s}_a \right) - q_b{}^c \bar{s}^d K_{cd} - \frac{1}{\alpha} \beta^a \bar{s}_a \hat{\nabla}_b \ln(\alpha) , \\
&= -\bar{B} q_b{}^c \left[-\frac{\nabla_c \alpha}{\alpha^2 \bar{B}} \beta^a \bar{s}_a - \frac{\nabla_c \bar{B}}{\alpha \bar{B}^2} \beta^a \bar{s}_a + \frac{1}{\alpha \bar{B}} (\beta^a \bar{s}_a) \right] \\
&\quad - q_b{}^c \bar{s}^d K_{cd} - \frac{1}{\alpha^2} \beta^a \bar{s}_a \hat{\nabla}_b \alpha , \\
&= \frac{1}{\alpha} \beta^a \bar{s}_a \hat{\nabla}_b \ln(\bar{B}) - \frac{1}{\alpha} \hat{\nabla}_b (\beta^a \bar{s}_a) - q_b{}^c \bar{s}^d K_{cd} , \\
&= \frac{\beta_\perp}{\alpha} \hat{\nabla}_b \ln(\bar{B}) - \frac{1}{\alpha} \hat{\nabla}_b \beta_\perp - q_b{}^c \bar{s}^d K_{cd} . \tag{8.1.3}
\end{aligned}$$

where we made the definition $\beta_\perp = \beta^a \bar{s}_a$. Putting all of the above expressions together, we find that

$$\begin{aligned}\nabla_a \bar{s}_b = & -H_{ab} - \bar{s}_a \hat{\nabla}_b \ln(\bar{B}) - n_b \bar{s}^c K_{ac} - n_a n_b \bar{s}^c \bar{\nabla}_c \ln(\alpha) \\ & + \frac{\beta_\perp}{\alpha} n_a \hat{\nabla}_b \ln(\bar{B}) - \frac{1}{\alpha} n_a \hat{\nabla}_b \beta_\perp - n_a q_b^c \bar{s}^d K_{cd}.\end{aligned}\quad (8.1.4)$$

So, this allows us to completely decompose $\nabla_a \bar{s}_b$ into ADM variables, while equation (3.1.16) enables us to decompose $\nabla_a n_b$ into ADM variables. We can now use this formalism to express the intrinsic horizon quantities in terms of ADM variables. Before fully doing this, we follow the notation of [2]³ and decompose the extrinsic curvature into components normal and tangential to the horizon:

$$\begin{aligned}J_{ab} &\equiv q_a^c q_b^d K_{cd} & J_a &\equiv q_a^b \bar{s}^c K_{bc} & K_\perp &\equiv \bar{s}^a \bar{s}^b K_{ab}, \\ K_{ab} &= \gamma_a^c \gamma_b^d K_{cd} = (q_a^c + \bar{s}_a \bar{s}^c) (q_b^d + \bar{s}_b \bar{s}^d) K_{cd}, \\ &= J_{ab} + J_a \bar{s}_b + \bar{s}_a J_b + \bar{s}_a \bar{s}_b K_\perp.\end{aligned}\quad (8.1.5)$$

Now, we are ready to work through the relevant intrinsic and extrinsic quantities. We start with the expansions, as they are the easiest.

³Though it should be noted that Cook does not define K_\perp and instead just writes it as $K - J$. Also, note that Cook works under a gauge where $\bar{B} = \beta_\perp = \bar{s}_a \beta^a$. This is not consistent with a naïve decomposition of the induced geometry on the horizon from the 4-geometry, since if t is the time coordinate and r is the radial coordinate of the horizon, then $\bar{B} = \frac{1}{\sqrt{\gamma^{rr}}}$ and $\beta_\perp = \bar{B} \alpha^2 g^{tr}$, which are clearly two inequivalent expressions in the general case.

$$\begin{aligned}
\theta &= q^{ab} \nabla_a \ell_b = q^{ab} \nabla_a [\alpha (n_a + \bar{s}_a)] , \\
&= \alpha (q^{ab} \nabla_a n_b + q^{ab} \nabla_a \bar{s}_b) , \\
&= \alpha (-J - H) .
\end{aligned} \tag{8.1.6}$$

A nearly identical derivation will show that $\theta_{(k)} = \frac{1}{2\alpha} (H - J)$. Immediately, we can therefore see that, in 3+1 language, the Isolated/Dynamical horizon condition is a requirement that $H = -J$ and $J > 0$ on the horizon. We can perform a similar decomposition to the shear:

$$\begin{aligned}
\sigma_{ab} &= q_a^c q_b^d \nabla_c \ell_d - \frac{1}{2} q_{ab} \theta , \\
&= q_a^c q_b^d \nabla_c [\alpha (n_d + \bar{s}_d)] + \frac{1}{2} \alpha q_{ab} (J + H) , \\
&= \alpha (-J_{ab} - H_{ab}) + \frac{1}{2} \alpha q_{ab} (J + H) .
\end{aligned} \tag{8.1.7}$$

This then shows us that the isolated horizon condition that σ_{ab} vanish is equivalent to the condition that $H_{ab} = -J_{ab}$. Tracing this equation will then automatically give you the $H = -J$ requirement for $\theta = 0$. Note that the twist $\omega_{ab} \equiv q_a^c q_b^d \nabla_{[c} \ell_{d]}$ is automatically zero. This should not be surprising since the 2-dimensional horizon is smoothly embedded in both the 3-surface and the 4-surface, and therefore must have a vanishing twist by

Frobenius's theorem⁴. Alternately, it is easy to substitute the definitions of n_a and \bar{s}_a in terms of gradients of functions and then take both of their twists, and then show that the twists are proportional to $\nabla_{[a}\nabla_{b]}f$ for some function in both cases, and are therefore zero. The Raychaudhuri equation (6.2.2) is then satisfied if, in addition to $J_{ab} = -H_{ab}$, we also have $T_{ab}\ell^a\ell^b = \alpha^2 (T_{ab}n^an^b + 2T_{ab}n^a\bar{s}^b + T_{ab}\bar{s}^a\bar{s}^b) = \alpha^2 (\rho - 2j^a\bar{s}_a + S_{ab}\bar{s}^a\bar{s}^b) = 0$. These conditions are then sufficient to ensure that the horizon is a WIH [16] as defined in equation (7.0.10).

8.1.1 Dynamical Horizon case

Now, let us consider the case of a dynamical horizon (DH). These horizons are still defined as a topologically compact level surface of $\theta = 0$. If the horizon is expanding, however, it is necessarily the case that outgoing null vectors can only momentarily have zero expansion relative to infinity, because the expansion of the horizon will make the old horizon radius lie in the interior of the horizon at later times. In other words, the light ray hovering stationary on the horizon at time t will be falling inward toward $r = 0$ at time $t + \delta t$. Therefore, the outgoing null vector is no longer a tangent to the Dynamical Horizon, which goes from being a null surface to a spacelike surface⁵. By this

⁴See Appendix B.3 in [13], amongst other sources

⁵Contracting horizons will be timelike surfaces. They have several properties, however, that are not horizon-like. In particular, timelike $\theta = 0$ surfaces will not be trapping surfaces—some null and timelike objects from inside these sorts of horizons will be able to escape to infinity. Furthermore, their existence necessarily requires the presence of matter violating the dominant energy condition. This may in fact be the case for black holes emitting Hawking radiation, but is otherwise not considered physically realistic. Therefore, we will not consider these surfaces here.

argument, then, the formerly null generators of the dynamical horizon must now be spacelike vectors. It is logical to adjust these generators in such a way as to make them as close to the ℓ^a as possible. We don't expect there to be dependence on the angular variables on the horizon, so the most logical choice for the generating vector v^a is given by the ansatz:

$$v^a = \ell^a + \psi \bar{s}^a. \quad (8.1.8)$$

It is easy to show that $v_a v^a = 2\alpha\psi + \psi^2$, which is positive so long as $-2\alpha < \psi$. So, therefore, for positive ψ , v^a is a spacelike vector that corresponds to ℓ^a for $\psi = 0$. Now, we enforce the condition that v^a generates a surface of $\theta = 0$:

$$\begin{aligned} \mathcal{L}_v \theta = 0 &= v^c \nabla_c \theta = (\ell^c + \psi \bar{s}^c) \nabla_c \theta, \\ &= \mathcal{L}_\ell \theta + \psi \bar{s}^c \nabla_c (q^{ab} \nabla_a \ell_b), \\ \frac{1}{\psi} (\mathcal{L}_v \theta - \mathcal{L}_\ell \theta) &= \bar{s}^c \nabla_c [-\alpha (J + H)], \\ &= -\bar{s}^c (J + H) \bar{\nabla}_c \alpha - \alpha \bar{s}^c \bar{\nabla}_c J - \alpha \bar{s}^c \bar{\nabla}_c H, \\ &= \bar{s}^c \theta \bar{\nabla}_c \ln(\alpha) - \alpha \bar{s}^c \bar{\nabla}_c J - \alpha \bar{s}^c \bar{\nabla}_c H. \end{aligned} \quad (8.1.9)$$

Further simplification of the second and third terms on the right hand side above is involved enough that each term will be dealt with individually. Before we begin, recall the Gauss-Codazzi condition from equation (3.1.21) tells us, in the case of a 2-surface embedded in a 3-space, we have, after using

the Hamiltonian constraint (3.1.23):

$$\begin{aligned}
\hat{R} &= \bar{R} - 2\bar{s}^a \bar{s}^b \bar{R}_{ab} - H^{ab} H_{ab} + H^2, \\
2\bar{s}^a \bar{s}^b \bar{R}_{ab} &= \bar{R} - \hat{R} - H^{ab} H_{ab} + H^2, \\
&= (16\pi\rho + K^{ab} K_{ab} - K^2) - \hat{R} - H^{ab} H_{ab} + H^2, \\
&= 16\pi\rho + J^{ab} J_{ab} + 2J^a J_a + K_\perp^2 - (K_\perp + J)^2 - \hat{R} - H^{ab} H_{ab} + H^2, \\
\bar{s}^a \bar{s}^b \bar{R}_{ab} &= \frac{1}{2} \left(16\pi\rho + J^{ab} J_{ab} - H^{ab} H_{ab} + 2J^a J_a - J^2 + H^2 - 2K_\perp J - \hat{R} \right).
\end{aligned} \tag{8.1.10}$$

Now, knowing this identity, we work toward decomposing $\bar{s}^a \bar{\nabla}_a H$:

$$\begin{aligned}
\bar{s}^a \bar{\nabla}_a H &= -\bar{s}^c \bar{\nabla}_c (q^{ab} \bar{\nabla}_a \bar{s}_b), \\
&= -\bar{s}^c q^{ab} \bar{\nabla}_c \bar{\nabla}_a \bar{s}_b - \bar{s}^c (\bar{\nabla}_c q^{ab}) \bar{\nabla}_a \bar{s}_b, \\
&= -\bar{s}^c q^{ab} \bar{R}_{cab}{}^d \bar{s}_d - \bar{s}^c q^{ab} \bar{\nabla}_a \bar{\nabla}_c \bar{s}_b + \bar{s}^a \bar{s}^c (\bar{\nabla}_a \bar{s}^b) \bar{\nabla}_c \bar{s}_b \\
&\quad + \bar{s}^c \bar{s}^b (\bar{\nabla}_c \bar{s}^a) \bar{\nabla}_a \bar{s}_b, \\
&= \bar{s}^c \bar{s}^d \bar{R}_{cd} - q^{ab} \bar{\nabla}_a (\bar{s}^c \bar{\nabla}_c \bar{s}_b) + q^{ab} (\bar{\nabla}_a \bar{s}^c) \bar{\nabla}_c \bar{s}_b \\
&\quad + \left[\hat{\nabla}^b \ln(\bar{B}) \right] \hat{\nabla}_b \ln(\bar{B}), \\
&= \frac{1}{2} \left(16\pi\rho + J^{ab} J_{ab} - H^{ab} H_{ab} + 2J^a J_a - J^2 + H^2 - 2K_\perp J - \hat{R} \right) \\
&\quad + q^{ab} \bar{\nabla}_a \left[\hat{\nabla}_b \ln(\bar{B}) \right] + H^{ab} H_{ab} + \frac{1}{\bar{B}^2} \left(\hat{\nabla}^b \bar{B} \right) \hat{\nabla}_b \bar{B}, \\
&= 8\pi\rho + \frac{1}{2} \left(J^{ab} J_{ab} + H^{ab} H_{ab} + H^2 - J^2 - \hat{R} \right) + J^a J_a - K_\perp J
\end{aligned}$$

$$+ \frac{1}{\bar{B}} \hat{\nabla}^2 \bar{B}. \quad (8.1.11)$$

This takes care of the third term in equation (8.1.9). In order to take care of the second term, we first start with the momentum constraint, equation (3.1.26), and contract it onto \bar{s}_a ⁶:

$$\begin{aligned} 8\pi j^a \bar{s}_a &= \bar{s}_a \bar{\nabla}_b (K^{ab} - \gamma^{ab} K) , \\ &= \bar{s}_a \bar{\nabla}_b (J^{ab} + \bar{s}^a J^a + \bar{s}^b J^a + \bar{s}^a \bar{s}^b K_\perp) - \bar{s}^a \bar{\nabla}_a (q^{bc} K_{bc} + \bar{s}^b \bar{s}^c K_{bc}) , \\ &= -J^{ab} \bar{\nabla}_b \bar{s}_a + \bar{\nabla}_b J^b + \bar{s}_a \bar{s}^b \bar{\nabla}_b J^a + K_\perp \bar{\nabla}_b \bar{s}^b + \bar{s}^b \bar{\nabla}_b K_\perp \\ &\quad - \bar{s}^a \bar{\nabla}_a J - \bar{s}^a \bar{\nabla}_a K_\perp , \\ &= J^{ab} H_{ab} + \hat{\nabla}_a J^a + 2\bar{s}^a \bar{s}^b \bar{\nabla}_a J_b - K_\perp H - \bar{s}^a \bar{\nabla}_a J , \\ \bar{s}^a \bar{\nabla}_a J &= J^{ab} H_{ab} + \hat{\nabla}_a J^a - 2\bar{s}^a J^b \bar{\nabla}_a \bar{s}_b - K_\perp H - 8\pi j^a \bar{s}_a , \\ &= J^{ab} H_{ab} + \hat{\nabla}_a J^a + 2J^b \hat{\nabla}_b \ln(\bar{B}) - K_\perp H - 8\pi j^a \bar{s}_a . \end{aligned} \quad (8.1.12)$$

Therefore, putting equations (8.1.12) and (8.1.11) into equation (8.1.9), we get

$$\begin{aligned} \frac{1}{\psi} (\mathcal{L}_v \theta - \mathcal{L}_\ell \theta) &= \bar{s}^c \theta \bar{\nabla}_c \ln(\alpha) - \alpha \left[\left(J^{ab} H_{ab} + \hat{\nabla}_a J^a + 2J^b \hat{\nabla}_b \ln(\bar{B}) - K_\perp H \right. \right. \\ &\quad \left. \left. - 8\pi j^a \bar{s}_a \right) + \left(8\pi \rho + \frac{1}{2} \left(J^{ab} J_{ab} + H^{ab} H_{ab} + H^2 - J^2 - \hat{R} \right) \right) \right] \end{aligned}$$

⁶Here, we make use of the decomposition in equation (8.1.5), and drop any terms that vanish either by the fact that two tensors are normal to each other or that vanish by virtue of \bar{s}_a having a fixed norm.

$$\begin{aligned}
& + J^a J_a - K_\perp J + \frac{1}{\bar{B}} \hat{\nabla}^2 \bar{B} \Big) \Big] , \\
& = \bar{s}^c \theta \bar{\nabla}_c \ln(\alpha) - \alpha \left[\frac{1}{2} (J^{ab} + H^{ab}) (J_{ab} + H_{ab}) - K_\perp (J + H) \right. \\
& \quad + \frac{1}{2} (H + J) (H - J) + \hat{\nabla}_a J^a + J^a J_a - \frac{1}{2} \hat{R} + 8\pi (\rho - j^a \bar{s}_a) \\
& \quad \left. + 2 J^b \hat{\nabla}_b \ln(\bar{B}) + \frac{1}{\bar{B}} \hat{\nabla}^2 \bar{B} \right] , \\
& = \bar{s}^c \theta \bar{\nabla}_c \ln(\alpha) - \alpha \left[\frac{1}{2\alpha^2} \left(\sigma^{ab} \sigma_{ab} + \frac{1}{2} \theta^2 \right) + \frac{1}{\alpha} K_\perp \theta - \theta \theta_{(k)} \right. \\
& \quad - \frac{1}{2} \hat{R} + \hat{\nabla}_a J^a + J^a J_a + 8\pi (\rho - \bar{s}^a j_a) + 2 J^a \hat{\nabla}_a \ln(\bar{B}) \\
& \quad \left. + \frac{1}{\bar{B}} \hat{\nabla}^2 \bar{B} \right] . \tag{8.1.13}
\end{aligned}$$

And finally, requiring that \mathcal{L}_v vanish, and that the surface is a dynamical horizon where the outgoing expansion θ vanishes, we get

$$\begin{aligned}
\psi &= \frac{\mathcal{L}_\ell \theta}{\alpha \left[\frac{1}{2\alpha^2} (\sigma^{ab} \sigma_{ab}) - \frac{1}{2} \hat{R} + \hat{\nabla}_a J^a + J^a J_a + 8\pi (\rho - j_a \bar{s}^a) + 2 J^a \hat{\nabla}_a \ln(\bar{B}) + \frac{1}{\bar{B}} \hat{\nabla}^2 \bar{B} \right]} , \\
&= \frac{-8\pi T_{ab} \ell^a \ell^b - \sigma_{ab} \sigma^{ab}}{\alpha \left[\frac{1}{2\alpha^2} (\sigma^{ab} \sigma_{ab}) - \frac{1}{2} \hat{R} + \hat{\nabla}_a J^a + J^a J_a + 8\pi (\rho - j_a \bar{s}^a) + 2 J^a \hat{\nabla}_a \ln(\bar{B}) + \frac{1}{\bar{B}} \hat{\nabla}^2 \bar{B} \right]} . \tag{8.1.14}
\end{aligned}$$

Several things should be noted about the above equation. The first thing is that ψ has a strong dependence on angular variations on the horizon. The traditional definition of the horizon angular momentum [11] [1] [17] is given by $L = \frac{1}{8\pi} \oint d^2x \sqrt{q} K_{ab} \bar{r}^a \hat{\phi}^a = \frac{1}{8\pi} \oint d^2x \sqrt{q} J_a \hat{\phi}^a$. Secondly, note that the only derivatives of the 2+1 ‘lapse’ function \bar{B} appearing in (8.1.14) are

2-dimensional derivatives—apparently ψ is not sensitive to radial dependencies in \bar{B} ⁷.

Finally, one might be concerned by the presence of \hat{R} in the denominator of this equation, since its value is heavily restricted by the topology of the horizon by the Gauss-Bonnet theorem. But, in fact, the presence of \hat{R} is actually quite fortunate, as in the case of the Schwarzschild metric, it is the only term in equation (8.1.14) that is not explicitly zero, and therefore, without the presence of \hat{R} , we would have no control over ψ approaching zero as our spacetime approached the spherically symmetric, non-dynamical case.⁸

8.1.2 Area Balance Law

We now show how this formalism can generate an area balance law, and in turn, how this area balance law dictates conditions upon the lapse function and shift vector, as originally proved by Ashtekar and Krishnan [16]. The key insight in this derivation is to note that the Dynamical Horizon is a spacelike 3-surface locally defined by the level surface $\theta = 0$ of the locally defined function θ . Therefore, it is a well-defined Cauchy surface that can be chosen for the 3+1 splitting of spacetime. Now, having made this definition, the next step is to calculate the mass flow across the horizon, and then to make what have now become commonplace 2+1 decompositions of the relevant geometrical

⁷This is something that we might expect, considering that \bar{B} is defined as $\frac{1}{\sqrt{\bar{\nabla}_i R \bar{\nabla}^i R}}$. Any radial variation in \bar{B} is gauge, as it can be adjusted merely by a reparameterization of the function R .

⁸For example, in the analysis of the Vaidya metric below, note that the $\dot{M} \rightarrow 0$ limit would be very poorly behaved without the 2-curvature term.

quantities. As we go through the following derivation, we will freely use the dynamical horizon condition $\theta = 0$ to eliminate any terms of the form $J + H$ ⁹:

$$\begin{aligned}
\int \sqrt{\gamma} d^3x 8\pi T_{ab} \ell^a n^b &= \int \sqrt{\gamma} d^3x \alpha \left(8\pi T_{ab} n^a n^b + 8\pi T_{ab} n^a \bar{s}^b \right), \\
&= \int \sqrt{\gamma} d^3x \alpha \left[\frac{1}{2} (\bar{R} - K^{ab} K_{ab} + K^2) \right. \\
&\quad \left. - \bar{s}_a \bar{\nabla}_b (K^{ab} - \gamma^{ab} K) \right] \\
&= \int \sqrt{\gamma} d^3x \alpha \left[\frac{1}{2} \left(\hat{R} + 2\bar{s}^a \bar{s}^b \bar{R}_{ab} + H^{ab} H_{ab} - H^2 - J^{ab} J_{ab} - 2J^a J_a \right. \right. \\
&\quad \left. - K_\perp^2 + K_\perp^2 + J^2 - 2K_\perp J \right) - \bar{s}_a \bar{\nabla}_b (J^{ab} + J^a \bar{s}^b + J^b \bar{s}^a + \bar{s}^a \bar{s}^b K_\perp) \\
&\quad \left. + \bar{s}^a \bar{\nabla}_a (J + K_\perp) \right], \\
&= \int \sqrt{\gamma} d^3x \alpha \left[\bar{s}^a \bar{s}^b \gamma^{cd} \bar{R}_{cadb} - K_\perp J - J_a J^a + \frac{1}{2} \left(\hat{R} + H^{ab} H_{ab} - J^{ab} J_{ab} \right) \right. \\
&\quad \left. - J^{ab} H_{ab} - \bar{s}_a \bar{s}^b \bar{\nabla}_b J^a - \bar{\nabla}_b J^b - \bar{s}^b \bar{\nabla}_b K_\perp - K_\perp H + \bar{s}^a \bar{\nabla}_a J + \bar{s}^a \bar{\nabla}_a K_\perp \right], \\
&= \int \sqrt{\gamma} d^3x \alpha \left[\bar{s}^a \bar{\nabla}_c \bar{\nabla}_a \bar{s}^c - \bar{s}^a \bar{\nabla}_a \bar{\nabla}_c \bar{s}^c - J^a J_a - \hat{\nabla}_a J^a \right. \\
&\quad \left. + \frac{1}{2} \left(\hat{R} + H_{ab} H^{ab} - J^{ab} J_{ab} - 2J^{ab} H_{ab} \right) - 2\bar{s}_a \bar{s}^b \bar{\nabla}_b J^a + \bar{s}^a \bar{\nabla}_a J \right], \\
&= \int \sqrt{\gamma} d^3x \alpha \left[\bar{\nabla}_c (\bar{s}^a \bar{\nabla}_a \bar{s}^c) + \bar{s}^c \bar{\nabla}_a H + (\bar{\nabla}_a \bar{s}^a)^2 - (\bar{\nabla}_a \bar{s}^c) \bar{\nabla}_c \bar{s}^a - J^a J_a \right. \\
&\quad \left. - \hat{\nabla}_a J^a + \frac{1}{2} \left(\hat{R} + H_{ab} H^{ab} - J_{ab} J^{ab} - 2J^{ab} H_{ab} \right) + 2\bar{s}^b J^a \bar{\nabla}_b \bar{s}_a \right]
\end{aligned}$$

⁹Note that the derivation to follow is done *intrinsically to the Dynamical Horizon*, using the constraint equations **on** the horizon. Therefore, it is best to think of the \bar{s}^a that appears below to be the unit normal parallel to the v^a discussed in this section, and for the unit timelike normal to be the timelike normal perpendicular to this—namely, the unit normal of a timelike observer falling into the black hole with no velocity transverse to the black hole surface.

$$\begin{aligned}
& + \bar{s}^a \nabla_a J] , \\
& = \int \sqrt{\gamma} d^3x \alpha \left[\bar{\nabla}_a (\bar{s}^b \bar{\nabla}_b \bar{s}^a) - J^a J_a - \hat{\nabla}_a J^a - 2J^a \hat{\nabla}_a \ln(\bar{B}) \right. \\
& \quad \left. + \frac{1}{2} \left(\hat{R} - H^{ab} H_{ab} - J^{ab} J_{ab} - 2J^{ab} H_{ab} \right) + \bar{s}^a \bar{\nabla}_a (J + H) \right] . \tag{8.1.15}
\end{aligned}$$

And now, we are nearly done. To further simplify this expression, we follow Ashtekar et al. [16], and make a few simplifying assumptions. First, we remember that \bar{s}^a is tangent to the DH. Therefore, $\bar{s}^a \bar{\nabla}_a (J + H) = 0$, and we can ignore the last term in (8.1.15). Second, we make a coordinate choice on the DH that makes \bar{B} depend only upon the radial coordinate. This eliminates the total divergence term as well as $\hat{\nabla}_a \bar{B}$. Finally, we make a choice of the lapse function so that $\alpha = \frac{1}{\bar{B}}$, which makes $\alpha \sqrt{\gamma} = \sqrt{q}$. This enables us to rewrite the 3-integral $\int d^3x$ as $\int dr \oint \sqrt{q} d^2x$. The integral of $\hat{\nabla}_a J^a$ becomes an integral of a total divergence over a boundary, while the integral of \hat{R} becomes the curvature invariant χ of the 2-dimensional section of the DH. Consequently, we can now rewrite (8.1.15) as

$$\begin{aligned}
\int d^3x 8\pi T_{ab} \ell^a n^b &= \int dr \oint \sqrt{q} d^2x \left[-J_a J^a + \frac{1}{2} \hat{R} - \frac{1}{2} \sigma^{ab} \sigma_{ab} \right] , \\
\chi \int dr &= \int d^3x (16\pi T_{ab} \ell^a n^b + 2J^a J_a + \sigma_{ab} \sigma^{ab}) \dots \tag{8.1.16}
\end{aligned}$$

The right hand side of 8.1.16 is known as the Hawking energy, which has several interesting properties:

First, if T_{ab} satisfies an energy condition such that $T_{ab} \ell^a n^b \geq 0$, as one

would expect for a fluid with a positive density greater in magnitude than its pressure, then the right hand side of the equation is explicitly positive-definite, which then immediately tells us that r must be increasing and that $\chi > 0$, meaning that the apparent horizons that make up the DH must individually have spherical topologies, and that the DH satisfies the law of nondecreasing area as one would expect from the second law of black hole dynamics.

Second, it is already apparent that (8.1.16) also contains the content of the first law of black hole dynamics. The left hand side term describes an increase in areal radius, which can be taken to be an infinitesimal area increase that can be identified with δA . The stress energy tensor term exactly describes the amount of mass entering the black hole, and thus can certainly be identified with δM . Meanwhile, the last two terms involve nondiagonal terms of K_{ab} , which are known to be associated with black hole angular momentum. Therefore, it shouldn't be very surprising that with some minor tweaks, (8.1.16) can be modified to produce a proof of the First Law of Black Hole dynamics, but with the advantage of being defined explicitly on the horizon, and that it is proved without the dependence on the details of the Kerr solution used in the proof given in (7.1.6).

8.2 Dynamics of the Vaidya solution

Now, let us apply this formalism to one of the simplest dynamical examples: the Vaidya solution [18]. This solution describes a spherically symmetric null dust in the background of a Schwarzschild black hole. In order to make the re-

relationship between the 3+1 split and these boundary conditions more explicit, we are going to explicitly provide a 3+1 split, and then compute the necessary value for ψ such that (8.1.9) is satisfied.

Start with the Vaidya metric as it is commonly given, where $M(v)$ is an arbitrary function of v :

$$ds^2 = - \left(1 - \frac{2 M(v)}{r} \right) dv^2 + 2 dv dr + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2. \quad (8.2.1)$$

While this form gives us the simplest and most direct description of the metric, it is not ideally suited for a 3+1 split, as a computation of the inverse metric tensor will show us that $dv_a dv_b g^{ab} = 0$. Therefore, we transform this to Kerr-like coordinates by making the coordinate transformation $v = t + r$. Upon completing this transformation, the metric tensor takes the form (in (t, r, θ, ϕ) coordinates):

$$g_{ab} = \begin{pmatrix} - \left(1 - \frac{2 M(t+r)}{r} \right) & \frac{2 M(t+r)}{r} & 0 & 0 \\ \frac{2 M(t+r)}{r} & \left(1 + \frac{2 M(t+r)}{r} \right) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix}. \quad (8.2.2)$$

Where M is now an arbitrary function of $t+r$, rather than v . This metric has the exact same form as the Schwarzschild metric does in Kerr coordinates. A labourious but straightforward computation gives that the Ricci scalar of

this metric is equal to zero, and that:

$$G_{ab} = R_{ab} = \frac{2\dot{M}}{r^2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (8.2.3)$$

Furthermore, we can compute two null vectors that have only t and r components, and after fixing their inner product with each other to be equal to -1 , we find that they are:

$$\begin{aligned} k_a dx^a &= \frac{1}{2c} (-dt - dr), \text{ and} \\ ell_a dx^a &= c \left[-dt \left(1 - \frac{2M}{r} \right) + dr \left(1 + \frac{2M}{r} \right) \right], \end{aligned} \quad (8.2.4)$$

where c is an as of now arbitrary constant. It is then easy to show that $q_{ab} = g_{ab} + \ell_a k_b + k_a \ell_b = r^2 d\Omega^2$, which then gives:

$$q_{ab} = \begin{pmatrix} -\left(1 - \frac{2M(t+r)}{r}\right) & \frac{2M(t+r)}{r} & 0 & 0 \\ \frac{2M(t+r)}{r} & \left(1 + \frac{2M(t+r)}{r}\right) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix}$$

$$\begin{aligned}
& + \begin{pmatrix} \left(1 - \frac{2M(t+r)}{r}\right) & -\frac{2M(t+r)}{r} & 0 & 0 \\ -\frac{2M(t+r)}{r} & -\left(1 + \frac{2M(t+r)}{r}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
& = r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2.
\end{aligned} \tag{8.2.5}$$

After doing this, we then can easily find the trace against q^{ab} the gradient of any one-form that has only temporal and radial dependence, since the contraction on q^{ab} will eliminate anything but angular terms, and the angular terms depend only on r :

$$\begin{aligned}
q^{ab} \nabla_a v_b &= q^{ab} \partial_a v_b - q^{ab} \Gamma_{ab}^c v_c, \\
&= 0 - \frac{1}{2} q^{ab} v^c (g_{ac,b} + g_{bc,a} - g_{ab,c}), \\
&= \frac{1}{2} q^{ab} v^c g_{ab,c}, \\
&= \frac{2}{r} v^r.
\end{aligned} \tag{8.2.6}$$

Now, we can use (8.2.6) to essentially read off the expansions of the two null vecors. The result is that $\theta_{(k)} = -\frac{1}{r_c}$ and $\theta = \frac{2c}{r} \left(1 - \frac{2M}{r}\right)$. Since, at the point where $r = 2M(t, r)$ we have $\theta_{(k)}$ negative and θ zero, it is clear that this surface represents a trapped surface. Note that it is also clearly dynamical, due to the explicit inclusion of the time coordinate above.

Now, we wish to 3+1 split this metric tensor. We will make the obvious

choice of $t = \text{constant}$ slices and proceed from there. With this choice, we clearly have $\alpha = \frac{1}{\sqrt{1+\frac{2M}{r}}}$, $\beta_a dx^a = \frac{2M}{r} dr$, and $n_a = (-\alpha, 0, 0, 0)$. The 3-metric tensor is given by:

$$\gamma_{ab} = \begin{pmatrix} 1 + \frac{2M}{r} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix}. \quad (8.2.7)$$

while the extrinsic curvature can be calculated according to the rule

$$\begin{aligned} K_{ab} &= -\gamma_a^c \nabla_c n_b, \\ &= -\left(\gamma_a^c \gamma_b^d \partial_c n_d - \gamma_a^d \gamma_b^e \Gamma_{de}^c n_c\right), \\ K_{ij} &= 0 - \Gamma_{ij}^t \alpha. \end{aligned} \quad (8.2.8)$$

which gives

$$\begin{aligned} K_{rr} &= \sqrt{1 + \frac{2M}{r}} \left(\frac{\dot{M}}{r} - \frac{4M\dot{M}}{r^2} - \frac{2M}{r^2} + \frac{6M^2}{r^3} \right), \\ K_{\theta\theta} &= \frac{K_{\phi\phi}}{\sin^2(\theta)} = \frac{2M}{\sqrt{1 + \frac{2M}{r}}}. \end{aligned} \quad (8.2.9)$$

With all other extrinsic curvature components equal to zero. We now 2+1 decompose onto the horizon. Our radial vector is given by $\bar{r}_a = \sqrt{1 + \frac{2M}{r}}(1, 0, 0)$, therefore giving us a spacelike “lapse” of $\sqrt{1 + \frac{2M}{r}}$. The 2-metric is simply the ordinary metric of a 2-sphere. Since K_{ab} only

has diagonal terms, we get $J_A = 0$, $K_\perp = \frac{1}{1+\frac{2M}{r}}K_{rr}$ and $J_{AB} = \frac{2M}{\sqrt{1+\frac{2M}{r}}}d\Omega^2$. Finally, we have the extrinsic curvature of the horizon in the 3-surface:

$$\begin{aligned}
H_{ab} &= -q_a^{c}q_b^{d}\bar{\nabla}_c\bar{r}_d, \\
H_{AB} &= \bar{\Gamma}_{AB}^{c}\bar{r}_c, \\
&= \frac{1}{2}\bar{r}^c(\gamma_{Ac,B} + \gamma_{Bc,A} - \gamma_{AB,c}), \\
&= \frac{1}{2}\frac{1}{\sqrt{1+\frac{2M}{r}}}\left(-\frac{\partial}{\partial r}\gamma_{AB}\right), \\
&= -\frac{r}{\sqrt{1+\frac{2M}{r}}}d\Omega^2.
\end{aligned} \tag{8.2.10}$$

Therefore, on the horizon, we have $H_{AB} = -J_{AB}$, which tells us, automatically, that $\sigma_{ab} = 0$ and $\theta = 0$, the latter fact we, of course, already derived in the full 4-space. The vanishing of the expansion and of the shear tremendously simplifies (8.1.13), which, on the horizon, now becomes:

$$\frac{1}{\psi}(\mathcal{L}_v\theta - \mathcal{L}_\ell\theta) = -\alpha\left[-\frac{1}{r^2} + 8\pi(\rho - j^a\bar{r}_a)\right]. \tag{8.2.11}$$

A glimpse at the Raychaudhuri equation (6.2.2), coupled with all of these computations, will show one that the only term remaining in $\mathcal{L}_\ell\theta$ is $-8\pi T_{ab}\ell^a\ell^b$. Setting $\mathcal{L}_v\theta = 0$, and solving for ψ ,

$$\psi = \frac{8\pi T_{ab}\ell^a\ell^b}{\alpha\left(\frac{1}{r^2} - 8\pi T_{ab}n^an^b - 8\pi T_{ab}n^a\bar{r}^b\right)}. \tag{8.2.12}$$

We now remember that $G_{ab} = \frac{2\dot{M}}{r^2} (dt + dr)^2$, which gives us $G_{ab}n^a n^b = \frac{\frac{2\dot{M}}{r^2}}{1 + \frac{2M}{r}}$, $G_{ab}n^a \bar{r}^b = -G_{ab}n^a n^b$ and $G_{ab}\ell^a \ell^b = \frac{8c\dot{M}}{r^2}$. Putting all of this together into (8.2.12), we get the result

$$\psi = \frac{8c\dot{M}}{\alpha}. \quad (8.2.13)$$

Now, all that is left is to set the value of c by requiring that $\ell_a = \alpha(n_a + \bar{r}_a)$, so that ℓ^a becomes the best approximation possible of the time evolution vector $t^a = \alpha n^a + \beta^a$ on the horizon. To do this, we simply calculate $\ell_a - \alpha n_a$, and then require that the answer be orthogonal to n^a

$$\begin{aligned} n_a(\ell^a - \alpha n^a) &= \ell_a n_b g^{ab} + \alpha, \\ 0 &= -\frac{c}{\sqrt{1 + \frac{2M}{r}}} \left[\left(1 + \frac{2M}{r}\right) \left(1 - \frac{2M}{r}\right) + \frac{2M}{r} \left(1 + \frac{2M}{r}\right) \right] \\ &\quad + \frac{1}{\sqrt{1 + \frac{2M}{r}}}, \\ &= -2c + 1 \longrightarrow c = \frac{1}{2}. \end{aligned} \quad (8.2.14)$$

So, after all of this, we find that the value ψ takes on the horizon is $4\dot{M}\sqrt{2}$. We can now use this quantity to calculate the time rate of change of the horizon's area:

First, we calculate the tangent vector to the dynamical horizon:

$$\begin{aligned}
v^a = \ell^a + \psi \bar{r}^a &= c \left[\left(1 + \frac{2M}{r} \right) \partial_t + \left(1 - \frac{2M}{r} \right) \partial_r \right] + \psi \left(\frac{1}{\sqrt{1 + \frac{2M}{r}}} \right) \partial_r, \\
&\hat{=} \partial_t + 4\dot{M} \partial_r.
\end{aligned} \tag{8.2.15}$$

where the hat indicates that the equality is only valid on the horizon.

Now, it is easy enough to directly calculate the rate of area increase of the black hole:

$$\begin{aligned}
\delta A &= \mathcal{L}_v \oint d^2x \sqrt{q}, \\
&= \oint d^2x \frac{1}{2} \sqrt{q} q^{ab} v^a \partial_a q_{ab}, \\
&= \oint d^2x 4\dot{M} \frac{2}{r} \sqrt{q}, \\
&= \oint d^2x 4\dot{M} \frac{2}{r} r^2 \sin \theta, \\
&= 16\pi \dot{M} (2M), \\
&= 32\pi \dot{M} M.
\end{aligned} \tag{8.2.16}$$

Which is a logical conclusion, considering that the straightforward, direct computation of the black hole area gives us the value $A = 4\pi r^2 = 16\pi M^2$, whose first derivative is clearly $32\pi M\dot{M}$. Clearly, this method was massive overkill for a situations such as the Vaidya metric with a known analytical solution. However, in a dynamical spacetime, where the location of the horizon

is not known, this technique gives an exact way to trace not only the horizon, but also the normals to the horizon as it expands.¹⁰

¹⁰Note that the area of the dynamical horizon is a slicing dependent quantity. See [\[19\]](#) for a direct analysis of the slicing dependence of the area of the Vaidya DH.

Appendix A

Null decomposition of the Minkowski and Kerr spacetimes

In order to further clarify the above procedure, here, we will work out a couple of examples of null decompositions. First, let us examine the Minkowski spacetime, given in spherical coordinates (t, r, θ, ϕ) :

$$g_{ab}dx^a dx^b = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2. \quad (\text{A.0.1})$$

We wish to examine the null geometry of the surface whose vector space is tangent to the outgoing vector $\ell^a = (1, 1, 0, 0)$. To do this, we define $v = -t + r$, which gives $g^{ab}dv_a = \ell^b$. Then, we do a coordinate transformation, replacing t with v , which gives the metric:

$$g_{ab} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix} \quad g^{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2(\theta)} \end{pmatrix}. \quad (\text{A.0.2})$$

Where the first row/column represents α , and the next three coordinates are (r, θ, ϕ) as they normally would be. Obviously, $d\alpha_a = (1, 0, 0, 0)$, and action upon this form with g^{ab} gives $\ell^a = (0, 1, 0, 0)$. We can see that, at any point in the spacetime, the tangent space is spanned by ℓ^a , $(0, 0, \frac{1}{r}, 0)$ and $(0, 0, 0, \frac{1}{r \sin(\theta)})$, along with one additional vector, normal to the second two vectors. We shall call this vector k^a , and a simple computation shows that there is only one choice of k^a that is null, normal to the two angular directions, and satisfies $k^a d\alpha_a = -1$. This choice is given by $k^a = (-1, -\frac{1}{2}, 0, 0)$, which, upon lowering with g_{ab} , is equivalent to $k_a = (\frac{1}{2}, -1, 0, 0)$

Now, we define the outgoing light cone at some time by setting $v = \text{constant}$, which means that $dv = 0$. Eliminating the appropriate columns from the above equation then gives the induced metric:

$$q_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix}. \quad (\text{A.0.3})$$

Which is manifestly degenerate. The general matrix q^{ab} satisfying $q_{am} q^{mn} q_{nb} =$

q_{ab} is given by:

$$q^{ab} = \begin{pmatrix} a & b & c \\ b & \frac{1}{r^2} & 0 \\ c & 0 & \frac{1}{r^2 \sin^2(\theta)} \end{pmatrix}. \quad (\text{A.0.4})$$

But, we now require that q^{ab} also satisfies $q^{ab} \underset{\leftarrow}{k}_a = 0$. By inspection, we can see that $\underset{\leftarrow}{k}_a = (-\frac{1}{2}, 0, 0)$. Therefore, we can see that $q^{ab} \underset{\leftarrow}{k}_a = -\frac{1}{2}(a, b, c)$, which then gives us:

$$q^{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2(\theta)} \end{pmatrix}. \quad (\text{A.0.5})$$

Which is, of course, the answer that one would have naïvely guessed to be the correct one. As before, one should carefully note that the the first index of the lowered induced metric acts upon what was the r index in the 4-dimensional spacetime. As shown above, this means that the corresponding vector is ℓ^a . Meanwhile, the raised indices also have an index that is labeled as r . This index, however, acts upon $\underset{\leftarrow}{k}_a$, and **not** upon ℓ_a , which has zero pullback onto the null surface. Note (as stated above) that this is, in fact necessary, since the covector space to a vector space is defined precisely by the condition that the covector basis e_a , when acting on the vector basis e^b must give the result δ_a^b , which is impossible if ℓ^a is a vector and if ℓ_a is a covector. The simplicity of the above Minkowski example was chosen to show

the inevitability of this fact.

Now, let us turn our attention the Kerr solution to the Einstein Equation. This solution is an algebraically special solution, and therefore, can be given in the form of a fiducial Minkowski metric plus one of the principal null vectors times itself. In spheroidal coordinates (t, r, ϕ, θ) , we have:

$$g_{ab} = \eta_{ab} + C k_a k_b \quad g^{ab} = \eta^{ab} - C k^a k^b. \quad (\text{A.0.6})$$

$$\eta_{ab} = \text{diag}(-1, \frac{B}{A}, A \sin^2 \theta, B) \quad k_a = (-\frac{A}{B}, -1, \frac{A \sin^2 \theta}{B}, 0). \quad (\text{A.0.7})$$

$$C = \frac{2MrB}{A^2} \quad A = r^2 + a^2 \quad B = r^2 + a^2 \cos^2 \theta. \quad (\text{A.0.8})$$

Where a and M are taken to be two parameters, with $|a| \leq M$. It is easy to check that k_a is null according to the Minkowski background metric, and therefore, it is also easy to verify that k_a is null relative to the full metric as well. Therefore, either η_{ab} or g_{ab} can be used to raise or lower indices on k_a , and we take η^{ab} to be the metric inverse of η_{ab} . We are now going to investigate the properties of this spacetime's horizon, and the properties of its embedding into the full 4-dimensional spacetime. Before we do so, it is useful to introduce the other principal null vector of this spacetime:

$$\ell_a = \left(-\left(\frac{A-2Mr}{2A}\right), \frac{B(A+2Mr)}{2A^2}, \frac{(A-2Mr)a\sin^2\theta}{2A}, 0 \right). \quad (\text{A.0.9})$$

Which, when raised using g^{ab} gives:

$$\ell^a = \left(\frac{A+2Mr}{2A}, \frac{A-2Mr}{2A}, \frac{(A+2Mr)a}{2A^2}, 0 \right). \quad (\text{A.0.10})$$

It can be verified that this vector has zero norm, and that $\ell_a k^a = \ell^a k_a = -1$. Now, note that this vector seems to have some quite strange behaviour when $A - 2Mr = 0$. Noting from the definition above that $A = r^2 + a^2$, it is easy enough to prove that $r^2 - 2Mr + a^2 = 0$ is satisfied when:

$$r = r_{\pm} \equiv M \pm \sqrt{M^2 - a^2}. \quad (\text{A.0.11})$$

Note that, for these values of r , $\ell_a \propto dr$, while ℓ^a has a vanishing coordinate in the ∂_r direction. We therefore expect that for these values of r , this is, in fact, a null submanifold with associated null vector ℓ^a . Using the above condition, and taking $r = r_+$ as the associated function (and thereby setting $dr = 0$), we take the 4-metric, do some algebra, and find the following induced metric:

$$q_{ab} = \begin{pmatrix} \frac{a^2 \sin^2\theta}{B_+} & \frac{-A_+ a \sin^2\theta}{B_+} & 0 \\ \frac{-A_+ a \sin^2\theta}{B_+} & \frac{A_+^2 \sin^2\theta}{B_+} & 0 \\ 0 & 0 & B_+ \end{pmatrix}. \quad (\text{A.0.12})$$

Where a subscript of $+$ indicates that a function of r is taking on the value $r = r_+$ in its r argument. This is a more complicated looking metric than the one that we derived for the Minkowski spacetime, but, once again, it is easy to verify that it has vanishing determinant, is nonnegative definite, and that its zero eigenvector is $(1, \frac{a}{A_+}, 0) = (\ell^t_+, \ell^\phi_+, \ell^\theta_+)$, and, since $\ell^r_+ = 0$, we can therefore just project this vector onto our null space and call it ℓ^a . We therefore have a null tangent space to our horizon spanned by ℓ^a and two spacelike angular directions. Finally, we pull back k_a on the horizon with result $k_a = (-\frac{A_+}{B_+}, \frac{A_+ \sin^2 \theta}{B_+}, 0)$.

Solving the equation $q_{am} q^{mn} q_{nb} = q_{ab}$ gives the following matrix, for undetermined functions X, Y, Z :

$$q^{ab} = \begin{pmatrix} \frac{B_+ + 2A_+ a(\sin^2 \theta)X - A^2(\sin^2 \theta)Y}{a^2 \sin^2 \theta} & X & Z \\ X & Y & (\frac{a}{A}) Z \\ Z & (\frac{a}{A}) Z & \frac{1}{B_+} \end{pmatrix}. \quad (\text{A.0.13})$$

Contracting this onto k_a and requiring that the answer be the zero vector then gives us the solution:

$$X = \frac{a}{B_+} \quad Y = \frac{1}{B_+ \sin^2 \theta} \quad Z = 0. \quad (\text{A.0.14})$$

Substituting this answer into the above expression for q^{ab} gives us:

$$q^{ab} = \begin{pmatrix} \frac{a^2 \sin^2 \theta}{B_+} & \frac{a}{B_+} & 0 \\ \frac{a}{B_+} & \frac{1}{B_+ \sin^2 \theta} & 0 \\ 0 & 0 & \frac{1}{B_+} \end{pmatrix}. \quad (\text{A.0.15})$$

Which we can then take to be the inverse 3-metric on the Kerr Horizon. As a final note, it is often easier to work with the so-called “untwisted” coordinates on the horizon. These coordinates are defined by the coordinate transformation:

$$\phi = \varphi + \frac{at}{A_+}. \quad (\text{A.0.16})$$

Under this transformation, we can follow the above procedure, and obtain the following 3-metric (in (t, φ, θ) coordinates):

$$q_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{A_+^2 \sin^2 \theta}{B_+} & 0 \\ 0 & 0 & B_+ \end{pmatrix} \quad q^{ab} = \begin{pmatrix} \frac{a^2 \sin^2 \theta}{B_+} & \frac{a}{A_+} & 0 \\ \frac{a}{A_+} & \frac{B_+}{A_+^2 \sin^2 \theta} & 0 \\ 0 & 0 & \frac{1}{B_+} \end{pmatrix}. \quad (\text{A.0.17})$$

As one might guess, this transformation makes $\ell^a = (1, 0, 0)$ while leaving k_a equal to the more complicated expression $(-1, \frac{A_+ a \sin^2 \theta}{B_+}, 0)$. Having done all of this work, it is now simple enough to proceed using quantities defined intrinsically on the Kerr Horizon. Note that the above procedures are non-trivial in the Kerr case—the inverse of the (θ, ϕ) part of q_{ab} is **not** equivalent to q^{ab} .

Now, having completed this Appendix, it should be clear how to go about inducing enveloping geometries onto null subspaces. In the end, the process is the same as for non-null subspaces, only with an extra step, because the tangent space and the cotangent space have to be treated separately.

Appendix B

Hamiltonian of Klein-Gordon Field in an external gravitational field

Here, I will decompose the action for a Klein-Gordon field coupled to an external gravitational field using a 3+1 formalism, and then derive the Hamiltonian and the associated equations of motion. In particular, we will show that while you can get boundary terms in the Hamiltonian, you get none of these boundary terms in the equations of motion. While it is not explicitly included here, the entire argument below is essentially unaltered if one were to add a self-interaction term $V(\phi)$ to the action, so long as it dependent only on the value of the field ϕ and none of its derivatives.

First, consider the Klein-Gordon action:

$$S = \int d^4x \sqrt{|g|} \mathcal{L} = \int d^4x \sqrt{|g|} \left(-\frac{1}{2} \nabla_a \phi \nabla^a \phi - \frac{1}{2} m^2 \phi^2 \right). \quad (\text{B.0.1})$$

Since we are concerning ourselves with only the dynamics of the scalar field in the background metric, and therefore holding the external metric fixed, and since we furthermore are not computing the stress-energy tensor, we will treat the square root of the determinant as part of the volume element, and treat the Lagrangian density as a true geometric density. This will simplify the calculation of boundary terms and does not change the content of any results. Therefore, from here on, the term involving the square root of the metric will be factored into the d^4x term, which will be understood to have the appropriate weight to be a volume element.

Knowing the Lagrangian density, it is easy to compute the momentum conjugate to the field:

$$\Pi \equiv \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \frac{\delta}{\delta \dot{\phi}} \left(-\frac{1}{2} \nabla_a \phi \nabla^a \phi - \frac{1}{2} m^2 \phi^2 \right) = -\nabla^t \phi. \quad (\text{B.0.2})$$

If this calculation were being done in Minkowski spacetime, then it would be direct to equate $\nabla^t \phi$ to $-\dot{\phi}$. Since we are instead in a 3+1 split, it is necessary to insert a factor of the inverse metric and decompose, yielding:

$$-\nabla^t \phi = -g^{ta} \nabla_a \phi = - \left(-\frac{1}{\alpha^2} \delta_t^a + \frac{1}{\alpha^2} \beta^i \right) \nabla_a \phi,$$

$$= \frac{1}{\alpha^2} \dot{\phi} - \frac{1}{\alpha^2} \beta^i \bar{\nabla}_i \phi. \quad (\text{B.0.3})$$

Where α is the lapse function and β^i is the shift vector for the particular slicing in question, $i, j, k \dots$ indicate 3-dimensional indices, and $\bar{\nabla}_i$ denotes the connection compatible with the 3-dimensional metric γ_{ij} . Putting this all together, we get:

$$\Pi = \frac{1}{\alpha^2} \left(\dot{\phi} - \beta^i \bar{\nabla}_i \phi \right). \quad (\text{B.0.4})$$

So, having done this, now we can see that:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \Pi \dot{\phi} - \frac{1}{2} \nabla^i \phi \nabla_i \phi - \frac{1}{2} m^2 \phi^2, \\ &= \frac{1}{2} \Pi \dot{\phi} - \frac{1}{2} g^{ai} \nabla_a \phi \nabla_i \phi - \frac{1}{2} m^2 \phi^2, \\ &= \frac{1}{2} \Pi \dot{\phi} - \frac{1}{2} g^{ti} \dot{\phi} \nabla_i \phi - \frac{1}{2} g^{ij} \nabla_i \phi \nabla_j \phi - \frac{1}{2} m^2 \phi^2, \\ &= \frac{1}{2} \Pi \dot{\phi} - \frac{1}{2\alpha^2} \beta^i \dot{\phi} \bar{\nabla}_i \phi - \frac{1}{2} \left(\gamma^{ij} - \frac{1}{\alpha^2} \beta^i \beta^j \right) \bar{\nabla}_i \phi \bar{\nabla}_j \phi - \frac{1}{2} m^2 \phi^2, \\ &= \frac{1}{2} \Pi \dot{\phi} - \frac{1}{2\alpha^2} (\beta^i \bar{\nabla}_i \phi) \dot{\phi} + \frac{1}{2\alpha^2} (\beta^i \bar{\nabla}_i \phi)^2 - \frac{1}{2} \bar{\nabla}^i \phi \bar{\nabla}_i \phi - \frac{1}{2} m^2 \phi^2. \end{aligned} \quad (\text{B.0.5})$$

Now, we can combine (B.0.5) with the definition of the Hamiltonian density $\mathcal{H} \equiv \Pi \dot{\phi} - \mathcal{L}$ to obtain the Hamiltonian density, and then we can use (B.0.2) to eliminate all dependence on $\dot{\phi}$:

$$\begin{aligned}
\mathcal{H} &= \Pi \dot{\phi} - \mathcal{L}, \\
&= \frac{1}{2} \Pi \dot{\phi} + \frac{1}{2\alpha^2} (\beta^i \bar{\nabla}_i \phi) \dot{\phi} - \frac{1}{2\alpha^2} (\beta^i \bar{\nabla}_i \phi)^2 + \frac{1}{2} \bar{\nabla}^i \phi \bar{\nabla}_i \phi + \frac{1}{2} m^2 \phi^2, \\
&= \frac{1}{2} \alpha^2 \Pi^2 + \frac{1}{2} \Pi \beta^i \bar{\nabla}_i \phi + \frac{1}{2} \Pi \beta^i \bar{\nabla}_i \phi + \frac{1}{2\alpha^2} (\beta^i \bar{\nabla}_i \phi)^2 - \frac{1}{2\alpha^2} (\beta^i \bar{\nabla}_i \phi)^2 \\
&\quad + \frac{1}{2} \bar{\nabla}^i \phi \bar{\nabla}_i \phi + \frac{1}{2} m^2 \phi^2, \\
&= \frac{\alpha^2}{2} \Pi^2 + \Pi (\beta^i \bar{\nabla}_i \phi) + \frac{1}{2} \bar{\nabla}^i \phi \bar{\nabla}_i \phi + \frac{1}{2} m^2 \phi^2.
\end{aligned} \tag{B.0.6}$$

This then immediately lets us integrate the Hamiltonian density to get the Hamiltonian (with the factor of $\sqrt{\gamma}$ once again an implied subcomponent of the d^3x :

$$H \equiv \int \mathcal{H} d^3x = \int d^3x \left(\frac{\alpha^2}{2} \Pi^2 + \Pi (\beta^i \bar{\nabla}_i \phi) + \frac{1}{2} \bar{\nabla}^i \phi \bar{\nabla}_i \phi + \frac{1}{2} m^2 \phi^2 \right). \tag{B.0.7}$$

Which gives the equivalent phase space action

$$S = \int d^4x \left[\Pi \dot{\phi} - \left(\frac{\alpha^2}{2} \Pi^2 + \Pi (\beta^i \bar{\nabla}_i \phi) + \frac{1}{2} \bar{\nabla}^i \phi \bar{\nabla}_i \phi + \frac{1}{2} m^2 \phi^2 \right) \right]. \tag{B.0.8}$$

And we therefore take (B.0.7) to define our Hamiltonian for this Klein-Gordon system. It is now trivial to get the equation of motion for ϕ :

$$\begin{aligned}
\frac{\delta}{\delta \Pi} S &= \frac{\delta}{\delta \Pi} \int d^4x \left[\Pi \dot{\phi} - \left(\frac{\alpha^2}{2} \Pi^2 + \Pi (\beta^i \bar{\nabla}_i \phi) + \frac{1}{2} \bar{\nabla}^i \phi \bar{\nabla}_i \phi + \frac{1}{2} m^2 \phi^2 \right) \right] , \\
&= \int d^4x \left[-\dot{\Pi} + \alpha^2 \Pi + \beta^i \bar{\nabla}_i \phi \right] .
\end{aligned} \tag{B.0.9}$$

Which, since the variation must be equal to zero under an arbitrary variation, gives us the equation of motion $\dot{\phi} = \alpha^2 \Pi + \beta^i \bar{\nabla}_i \phi$. The variation with respect to ϕ required to obtain the equation of motion for $\dot{\Pi}$ is somewhat more intricate however:

$$\begin{aligned}
\frac{\delta}{\delta \phi} S &= \int d^4x -\dot{\Pi} + (m^2 \phi - \bar{\nabla}^i \bar{\nabla}_i \phi) \\
&\quad + \frac{\delta}{\delta \phi} \int d^4x (\bar{\nabla}_i (\Pi \beta^i \phi) - \phi \Pi \bar{\nabla}_i \beta^i - \phi \beta^i \bar{\nabla}_i \Pi) , \\
0 + \int d^4x \dot{\Pi} &= \int d^4x (m^2 \phi - \bar{\nabla}^i \bar{\nabla}_i \phi - \Pi \bar{\nabla}_i \beta^i - \beta^i \bar{\nabla}_i \Pi) + \oint d^3x (r_i \Pi \beta^i) , \\
&= \int d^4x [m^2 \phi - \bar{\nabla}^i \bar{\nabla}_i \phi - \bar{\nabla}_i (\Pi \beta^i)] + \oint d^3x (r_i \Pi \beta^i) ,
\end{aligned} \tag{B.0.10}$$

$$= \int d^4x (m^2 \phi - \bar{\nabla}^i \bar{\nabla}_i \phi) . \tag{B.0.11}$$

Where, in going from (B.0.10) to (B.0.11) we noted that the divergence in the bulk integral precisely canceled the boundary integral via a simple application of Gauss's theorem. Therefore, we obtain the equation of motion for Π , which is simply $\dot{\Pi} = \bar{\nabla}^i \bar{\nabla}_i - m^2 \phi$. We have now found the complete

equations of motion for this Hamiltonian system after taking a variation of our dynamical variables, and importantly, we have found that there are no dangling boundary terms remaining after our variation has been taken, meaning that it is not necessary to add any counterterms to our original action—the variation of the phase space action will be zero for any (Π, ϕ) that satisfy the equations of motion for the Klein-Gordon field.

As a final step, it would be remiss to not check our result against the traditional empty-space Klein-Gordon equations of motion in Cartesian coordinates. This is simply equivalent to setting $\gamma^{ij} = \delta^{ij}$, and making the choices $\beta_i = 0$ and $\alpha = 1$. If we do this we have:

$$\dot{\phi} = \Pi \qquad \dot{\Pi} = \partial^i \partial_i \phi - m^2 \phi. \qquad (\text{B.0.12})$$

If we take a time derivative of the equation for $\dot{\phi}$, and then substitute this answer into the equation for $\dot{\Pi}$, we get

$$\begin{aligned} \ddot{\phi} &= \partial^i \partial_i \phi - m^2 \phi, \\ 0 &= -\ddot{\phi} + \partial^i \partial_i \phi - m^2 \phi, \\ 0 &= \eta^{ab} \partial_a \partial_b \phi - m^2 \phi. \end{aligned} \qquad (\text{B.0.13})$$

And (B.0.13), indeed, is the Klein-Gordon equation of motion.

Appendix C

Boundary terms in the Schwarzschild-Roberson-Walker spacetime

As another example, let us consider the case of the Schwarzschild-Robertson-Walker spacetime. This is furnished by a simple extension of both the Schwarzschild and Robertson-Walker spacetimes, and is given by the line element¹

$$g_{ab}dx^a dx^b = - \left(1 - \frac{2M}{r}\right) dt^2 + (a(t))^2 \left[\left(\frac{1}{1 - \frac{2M}{r}} \right) dr^2 + r^2 d\theta^2 + r^2 \sin(\theta) d\phi^2 \right] . \quad (\text{C.0.1})$$

¹Note: the function $a(t)$ is meant to be evocative of the Robertson-Walker function $a(t)$. It is **not** meant to have anything to do with the Kerr parameter a .

If we take the limit $a(t) = 1, \dot{a}(t) \rightarrow \ddot{a}(t) \rightarrow 0$ we clearly recover the Schwarzschild spacetime, while if we take the limit $M \rightarrow 0$ we recover the flat Robertson-Walker model. Furthermore, direct computation of the curvature generated by this line element shows that its Einstein tensor is:

$$G_{ab}dx^a dx^b = 3 \left(\frac{\dot{a}}{a} \right)^2 dt^2 + 2dt dr \left(\frac{2M}{r^2} \right) \left(\frac{\dot{a}}{a(1 - \frac{2M}{r})} \right) - dr^2 \left(\frac{\dot{a}^2 + 2a\ddot{a}}{(1 - \frac{2M}{r})^2} \right) - d\theta^2 r^2 \left(\frac{\dot{a}^2 + 2a\ddot{a}}{(1 - \frac{2M}{r})^2} \right) - d\phi^2 r^2 \sin^2(\theta) \left(\frac{\dot{a}^2 + 2a\ddot{a}}{(1 - \frac{2M}{r})^2} \right). \quad (\text{C.0.2})$$

Which manifestly satisfies $G_r^r = G_\theta^\theta = G_\phi^\phi = - \left(\frac{\dot{a}^2 + 2a\ddot{a}}{(1 - \frac{2M}{r})^2} \right)$. We can therefore interpret the line element given in equation (C.0.1) as the gravitational field of a mass M surrounded by a strain-free dynamical fluid with a radial current². Note that this solution, however, is likely unphysical, as computing the trace of the Einstein tensor above will show that both the Ricci scalar and $R_{ab}R^{ab}$ contain a singularity at what would otherwise be the black hole horizon at $r = 2M$. Assuming a polytropic equation of state for the fluid can produce a concrete form for the function $a(t)$, and given this form, numerical computation of the geodesic equations generated from equation (C.0.1) will show that this coordinate is accessible in a finite amount of proper time, so the black hole horizon in the McVittie solution is a naked singularity. Further-

²by changing the time coordinate to $\tau = 3\ln(a) + \ln(1 - \frac{2M}{r})$, we can diagonalize the Einstein tensor. This will make the fluid pressure in the r direction different from the fluid pressure in the angular directions as well as making the metric tensor nondiagonal. Therefore, the fluid is not isotropic in its comoving frame

more, computation of the scalar $C_{abcd}C^{abcd}$ out of the metric (C.0.1) produces a result that is also singular, making it unlikely that this singularity arises solely due to a caustic in the matter³.

Furthermore, varying the line element with respect to t will generate the equation:

$$\begin{aligned}\frac{dE_{Schwarzschild}}{dt} &= -a\dot{a} \left(\dot{r}^2 \frac{1}{1 - \frac{2M}{r}} + \dot{\theta}^2 r^2 + \dot{\phi}^2 r^2 \sin^2(\theta) \right), \\ &= -\frac{\dot{a}}{a} E_{Robertson-Walker}.\end{aligned}\tag{C.0.3}$$

Here $E_{Schwarzschild}$ is the formerly conserved quantity in the pure Schwarzschild spacetime arising from the former killing vector ∂_t , and $E_{Robertson-Walker}$ is the conserved energy in the pure Robertson-Walker spacetime that is derived by combining the geodesic equation for the time coordinate with the unit timelike condition.

In particular, this means that for an expanding (contracting) universe, geodesics will tend to have their energy redshifted (blueshifted) as time progresses. This dynamical effect will make stable circular orbits impossible, and on a timescale governed by the Hubble time and by the mass of the central object, initially circular orbits will tend to fall into the singularity at $r = 2M$ (or spiral outward toward infinity and become unbound). Numerical solutions

³There is one exception to this: if $a(t)$ satisfies $\frac{\dot{a}(t)}{a(t)} = -\frac{\dot{a}(t)}{\dot{a}(t)}$, which then makes $a(t)$ an exponential function. In this case, $C_{abcd}C^{abcd}$ and R become finite, although $R_{ab}R^{ab}$ retains its singularity. So, while the singularity remains, its strength is lessened and its character changed.

of the geodesic equations verify this property of geodesics and generalize it to elliptical orbits as well. A few plots of solutions with polytropic equations of state are plotted below. The geodesic equation was solved numerically using the Mathematica software package, which also produced the plots.

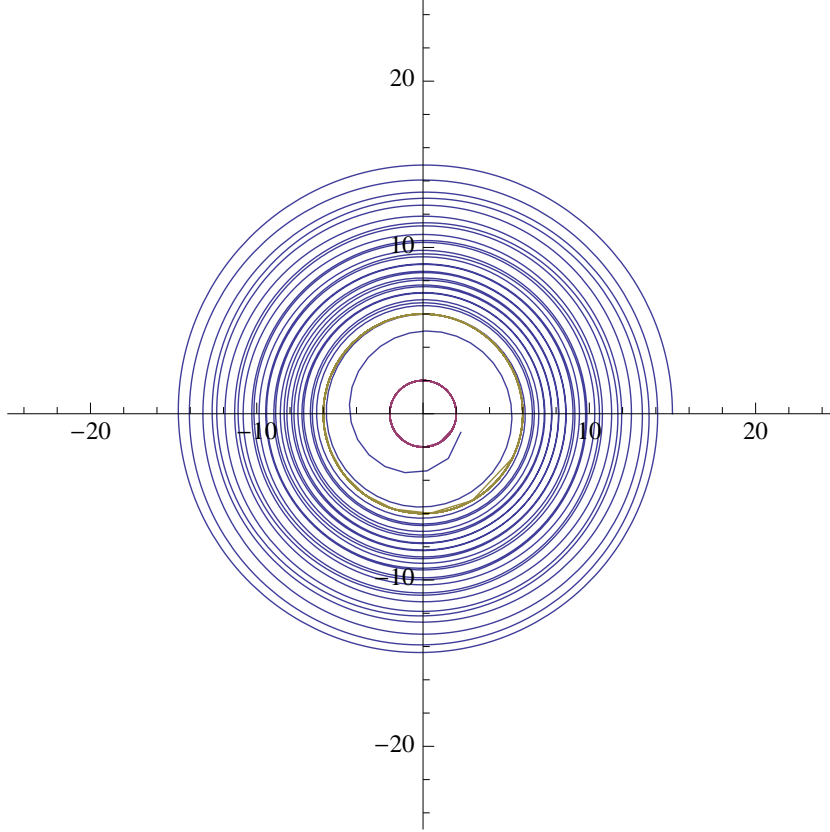


Figure C.1: A graph of the solution for an equatorial Geodesic with $r_0 = 15M$, $H_0 = \frac{5.41 \times 10^{-5}}{M}$, and $L = L_{circular}$. The red circle represents $r = 2M$, while the gold circle represents the innermost stable circular orbit at $r = 6M$. The test particle completes 21 full orbits before falling into the black hole.

Since a particular orbit is only bound for a finite amount of time in this spacetime, we therefore expect to find some interesting time-dependence in the energetics of this spacetime. In particular, this example was chosen as a way

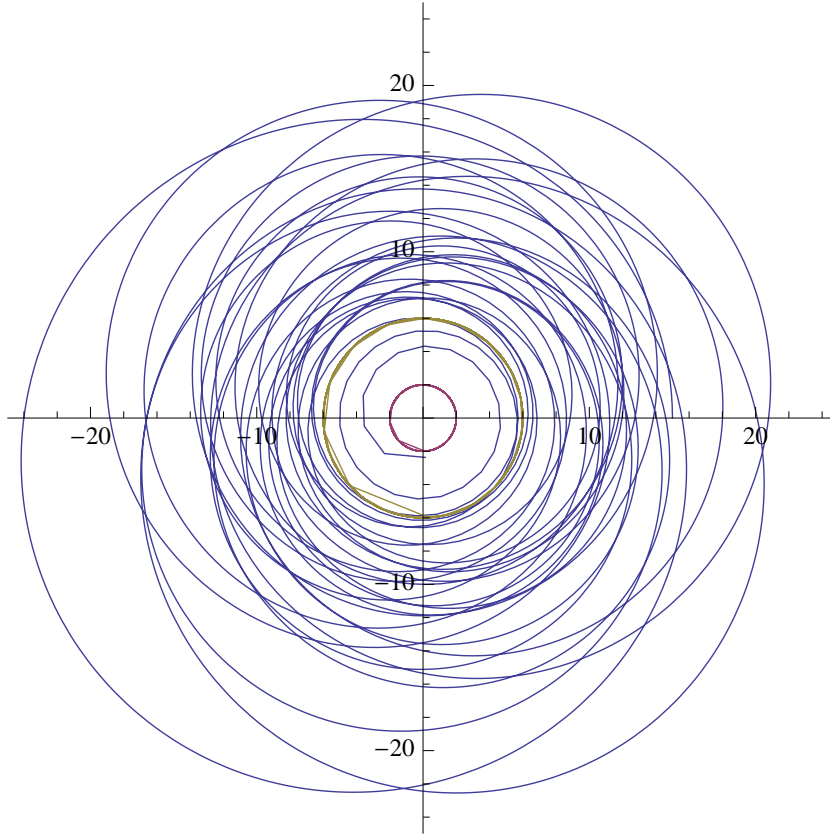


Figure C.2: A graph of the solution for an equatorial Geodesic with $r_0 = 15M$, $H_0 = \frac{5.41 \times 10^{-5}}{M}$, and $L = 1.1L_{circular}$. An addition of a relatively small amount of additional angular momentum than this produces a simple escape orbit, approximately at $L = 1.35M$. The red circle represents $r = 2M$, while the gold circle represents the innermost stable circular orbit at $r = 6M$. The test particle completes 21 full orbits before falling into the black hole. The test particle completes 28.9 full orbits before plunge happens

to show that it is, in fact possible to generate reasonable-seeming spacetimes that have time dependent ADM masses, as shall be shown below.

So now, let us compute this spacetime's ADM mass. For the slicing coordinate, we choose the coordinate t from the line element above. Meanwhile, for the fiducially flat comparison background metric $\tilde{\gamma}_{ij}$, we make the obvious

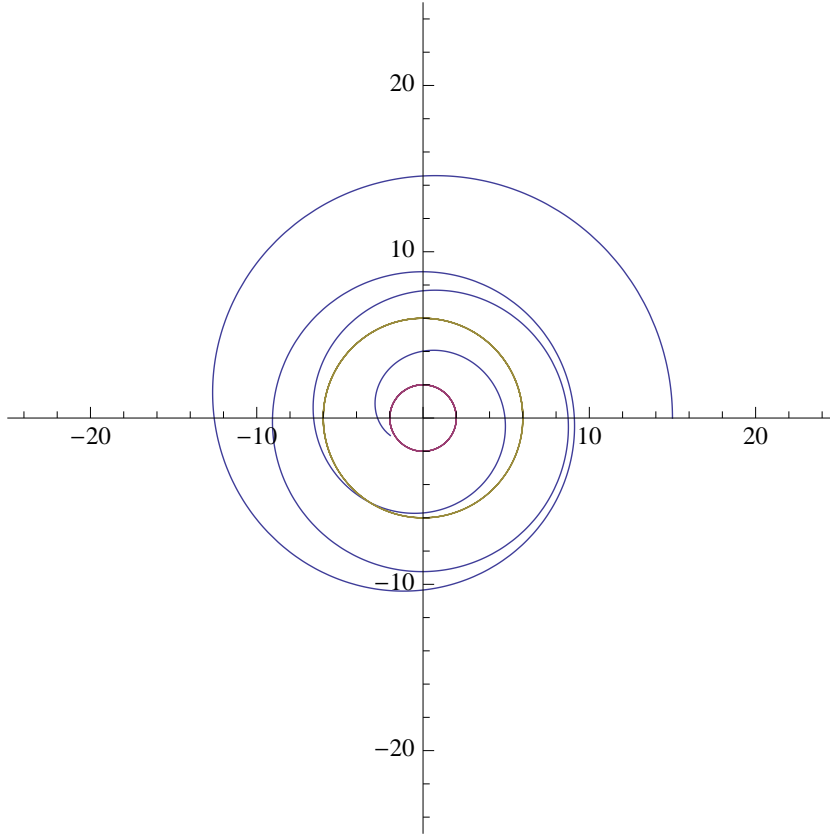


Figure C.3: A graph of the solution for an equatorial Geodesic with $r_0 = 15M$, $H_0 = \frac{5.41 \times 10^{-4}}{M}$, and $L = L_{\text{circular}}$. First, note that the Hubble constant for this plot is ten times the Hubble constant for the other two plots. The red circle represents $r = 2M$, while the gold circle represents the innermost stable circular orbit at $r = 6M$. The test particle completes 3 full orbits before falling into the black hole. Note the degree to which there is already a poor division between the inspiral phase and the plunge phase, although the geodesic does manage to complete three full orbits. Orbits with values of H_0 much larger than this tend to not even complete whole orbits before plunge, even when given large amounts of angular momentum.

choice of the spherical Euclidean metric given by choosing $a(t) = 1$, $M = 0$ ⁴.

⁴One might object to this choice, arguing that the spacetime is not, in fact, asymptotically flat. However, each timeslice under this choice IS asymptotically flat, and with a rescaling of the time coordinate, this spacetime is conformal to Schwarzschild spacetime, which is

Then, denoting covariant differentiation relative to $\tilde{\gamma}_{ij}$ as $\tilde{\nabla}_i$, we compute the ADM mass:

$$\begin{aligned}
16\pi M_{ADM} &= \oint \sqrt{\tilde{q}} \tilde{\gamma}^{ij} \tilde{r}^k \left(\tilde{\nabla}_i \gamma_{jk} - \tilde{\nabla}_k \gamma_{ij} \right), \\
&= \oint \sqrt{\tilde{q}} \tilde{\gamma}^{ij} \tilde{r}^k \left(\partial_i \gamma_{jk} - \partial_k \gamma_{ij} - \tilde{\Gamma}_{ij}^\ell \gamma_{\ell k} + \tilde{\Gamma}_{kj}^\ell \gamma_{\ell i} \right), \\
&= \oint \sqrt{\tilde{q}} \tilde{\gamma}^{ij} r^k (\partial_i \gamma_{jk} - \partial_k \gamma_{ij}) + \oint \sqrt{\tilde{q}} \tilde{\gamma}^{ij} r^k \left(\tilde{\Gamma}_{jk}^\ell \gamma_{\ell i} - \tilde{\Gamma}_{ij}^\ell \gamma_{\ell k} \right), \\
&= \oint \sqrt{\tilde{q}} (-\tilde{\gamma}^{\theta\theta} \partial_r \gamma_{\theta\theta} - \tilde{\gamma}^{\phi\phi} \partial_r \gamma_{\phi\phi}) + \oint \sqrt{\tilde{q}} \tilde{\gamma}^{ij} \left(\tilde{\Gamma}_{jr}^\ell \gamma_{\ell i} - \tilde{\Gamma}_{ij}^\ell \gamma_{\ell r} \right), \\
&= \oint d\phi d\theta r^2 \sin(\theta) \left(-\frac{4a^2}{r} \right) \\
&\quad + \oint \sqrt{\tilde{q}} \tilde{\gamma}^{ij} \left(\left(\frac{1}{r} \right) (\delta_j^\theta \gamma_{i\theta} + \delta_j^\phi \gamma_{i\phi}) - \frac{a^2}{1 - \frac{2M}{r}} \left(\frac{1}{r} \right) (-\tilde{q}_{ij}) \right), \\
&= \oint d\phi d\theta r^2 \sin(\theta) \left(-\frac{2a^2}{r} + \left(\frac{2a^2}{r} \right) \frac{1}{1 - \frac{2M}{r}} \right), \\
&= \oint d\phi d\theta r^2 \sin(\theta) \left(\frac{2a^2}{r} \right) \left(\frac{1}{1 - \frac{2M}{r}} \right) \left(\frac{2M}{r} - 1 + 1 \right), \\
&= \frac{4Ma^2}{1 - \frac{2M}{r}} \oint d\phi d\theta \sin(\theta), \\
&= \frac{16\pi Ma^2}{1 - \frac{2M}{r}}. \tag{C.0.4}
\end{aligned}$$

And then, when we take the limit $r \rightarrow \infty$, we obtain the result $M_{ADM} = Ma^2$, indicating that the ADM mass of this spacetime is time-dependent.

asymptotically flat. Also, since none of the derivatives in the ADM formula is a time derivative, and there are two factors of the lowered metric (inside the parentheses, and in the two-metric determinant), and one factor of the raised metric, this choice yields the same formula as you would obtain by choosing a fiducial metric by defining $M = 0$

Furthermore, this behaviour explains the odd behaviour of the geodesics of this spacetime—rather than interpreting the energy of an orbiting particle as being redshifted by the cosmological expansion, we can also interpret the mass concentrated in the central object increasing by an amount proportional to the scale factor over time. Therefore, initially circular orbital paths eventually find themselves attracted by a mass larger than the one they were attracted by initially, and see the radius of their orbit shrink. Eventually, they find themselves beyond the stable limit of their orbit, and have no choice but to plunge into the surface of the singularity.

The conclusion to reach is that while, for asymptotically flat spacetimes, the ADM mass is conserved, there do exist examples where one can find a time-dependence in the ADM quantities. Relativity’s insistence on local mass-energy conservation does not necessarily imply global mass-energy conservation. This result arises from the fact that this spacetime does not admit a global timelike Killing vector on the “sphere at infinity”⁵.

⁵In classical mechanics, it is not wholly abnormal for boundary conditions to change the overall value of the Hamiltonian, even if the bulk states stay the same. Consider a system of a string vibrating in its fundamental mode on a string of length L , mass per length μ and amplitude A . Now, a simple evaluation of the Hamiltonian of this system [20] will show you that the total energy in the string is given by $\frac{\pi^2 \mu A^2}{4L}$. Therefore, if one were to lengthen the string without changing its tension, and if \dot{L} were much smaller than the characteristic wave velocity in the string, one would expect that the string would stay in its new fundamental mode, and have energy $\frac{\pi^2 \mu A^2}{4L(t)}$. Thus the slow expansion of the string would allow the string to do work, causing it to lose energy. The proposal in this appendix is that we can similarly interpret time-dependent boundary terms in relativity—they are work terms indicating net *global* energy generated by the gravitational field, just as the factor expanding the classical string creates/removes net energy from that system.

Appendix D

On various foliations of a spacelike surface in a 3+1 split

Note that, when dealing with boundary conditions in 3+1 foliations, we often have to deal with TWO foliations of boundary terms—one involving the choice of a constant τ slice, giving a spacelike surface and one involving the choice of a constant R slice, which gives the boundary of that spacelike surface. In this appendix we will work out some consistency problems involving these two choices. The general scope of this argument is following that given in [21], but the notation chosen in this derivation is chosen so as to match the notation in the rest of this work, as certain results derived in this appendix will be used heavily elsewhere in this work, particularly in Chapters 3, 5, and 6. Also, certain details are worked out in a slightly different way. For the below derivation, unless otherwise noted, we will be using the inclusion operator to map intrinsic 2- and 3-geometries into the enveloping 4-geometry, since we will

be comparing geometric quantities to each other. Also, we will freely raise and lower vectors using the full 4-metric g_{ab} and its inverse.

Now, consider a 4-manifold with boundary \mathbb{M} with topology $\mathbb{R} \times \mathbf{m}$ with boundary $\mathbf{m}|_0 \cup \mathbf{m}|_f \cup (\mathbb{R} \times \partial\mathbf{m})$. We will label the \mathbb{R} portion of the spacetime with the function τ . Similarly, there is some function R such that R is constant on $\partial\mathbf{m}$ and such that R is not constant on some neighborhood of $\partial\mathbf{m}$. For now, we will assume that $\nabla_a R$ is a spacelike object, and examine the null case at the end of this appendix. Now, as was done in (2.1), we can easily define the two normals to this surface as they live in the enveloping 4-space. And from these normals, we can construct projection operators onto their respective normal spaces:

$$\begin{aligned} n_a &= \alpha \nabla_a \tau & s_a &= B \nabla_a R, \\ \alpha &= \frac{1}{\sqrt{|g^{ab} (\nabla_a \tau) \nabla_b \tau|}} & B &= \frac{1}{\sqrt{g^{ab} (\nabla_a R) \nabla_b R}}, \\ \gamma^{ab} &= g^{ab} + n^a n^b & \bar{\gamma}^{ab} &= g^{ab} - s^a s^b. \end{aligned} \tag{D.0.1}$$

Note, however, that it is **not** the case that $n^a s_a = 0$, since we have done nothing to guarantee that these things are normal to each other. In particular, if we choose τ and R as coordinates, so that our coordinate system is (τ, R, x^3, x^4) , it should be clear that it should not generally be the case that $g^{\tau R} = 0$ on any particular surface. Therefore, n^a and s^a are not a particularly good choice of basis for the normal vector space to $\partial\mathbf{m}$.

So, what to do? Clearly, it will be necessary to use the projection operators in a Gramm-Schmidt procedure. Since we are starting with the same 4-manifold and we are projecting down to the same final 2-manifold, it shouldn't matter which vector we start with and which vector we project. As we will see below, however, there is not complete equivalence¹. So, we can define two different projections of the vectors, first onto the barred space defined by γ^{ab} and second, onto the double barred space defined by $\bar{\gamma}^{ab}$. Finally, we define the projection operators defined by these two different methods, and note that they must be the same, as the target 2-space is simply $\partial\mathbf{m}$

$$\begin{aligned}\bar{s}_a &= \bar{B}\bar{\nabla}_a R = \bar{B}\gamma_a{}^b \nabla_b R & \bar{n}_a &= \bar{\alpha}\bar{\bar{\nabla}}_a \tau = \bar{\alpha}\bar{\gamma}_a{}^b \nabla_b \tau, \\ \bar{B} &= \frac{1}{\sqrt{\gamma^{ab}\bar{s}_a\bar{s}_b}} & \bar{\alpha} &= \frac{1}{\sqrt{|\bar{\gamma}^{ab}\bar{n}_a\bar{n}_b|}}.\end{aligned}\tag{D.0.2}$$

$$q^{ab} = g^{ab} + n^a n^b - \bar{s}^a \bar{s}^b = g^{ab} - s^a s^b + \bar{n}^a \bar{n}^b$$

Now, the obvious conclusion that we can get from equations (D.0.2) is that $n^a n^b - \bar{s}^a \bar{s}^b = -s^a s^b + \bar{n}^a \bar{n}^b$. We are going to use this fact in order to derive an interesting conclusion regarding the two sets of vectors. In what follows,

¹For an extreme example for why this should be the case, consider the Schwarzschild spacetime in Kerr-Schild coordinates at $r = 2M$. If we slice by Kerr Schild time first, then we get a spacelike 3-manifold that stretches all of the way to the singularity at $r = 0$. The $r = 2M$ surface appears like a perfectly well-behaved spacelike sphere. Null geometry would never have to be used or invoked.

Now, however, if we were to choose to slice the spacelike slice FIRST, we would get a null 3-surface, and only recover the spacelike 2-geometry after setting the null parameter of the horizon equal to a constant. Care must be taken in order to ensure consistency between these two approaches

note that the constructions of all of our projection operators are made in such a way that they annihilate relevant vectors. Therefore, $n^a \bar{s}_a = \bar{n}^a s_a = 0$. it is **not** the case, however, that the other contractions of these normals are going to vanish. Therefore, we will make the definition $n^a s_a = \psi$.

Expanding the definition of \bar{s}_a :

$$\begin{aligned}
\bar{s}_a &= \bar{B} \gamma_a^b \nabla_b R = \bar{B} (\delta_a^b + n_a n^b) \nabla_b R, \\
&= \bar{B} \nabla_a R + \bar{B} n_a n^b \nabla_b R, \\
&= \frac{\bar{B}}{B} (s_a + n_a \psi), \\
s_a &= \frac{B}{\bar{B}} \bar{s}_a - \psi n_a.
\end{aligned} \tag{D.0.3}$$

Similarly,

$$\begin{aligned}
\bar{n}_a &= \bar{\alpha} \bar{\gamma}_a^b \nabla_b \tau = \bar{\alpha} (\delta_a^b - s_a s^b) \nabla_b \tau, \\
&= \bar{\alpha} (\nabla_a \tau - s_a s^b \nabla_b \tau), \\
&= \frac{\bar{\alpha}}{\alpha} (n_a - s_a \psi), \\
&= \frac{\bar{\alpha}}{\alpha} n_a - \psi \frac{\bar{\alpha}}{\alpha} \left(\frac{B}{\bar{B}} \bar{s}_a - \psi n_a \right), \\
&= \frac{\bar{\alpha}}{\alpha} (1 + \psi^2) n_a - \left(\frac{\bar{\alpha} B}{\alpha \bar{B}} \right) \psi \bar{s}_a.
\end{aligned} \tag{D.0.4}$$

So, we have now solved for the reverse order ADM vectors \bar{s}_a and n_a in terms of their standard order counterparts, the “lapse” functions and the

parameter ψ . Next, we enforce the fact that s_a and \bar{n}^a is an orthonormal basis. Then, we will check to make sure that the consistency relationship derived from equating the q^{ab} is valid. Starting with equations (D.0.4) and (D.0.3), and assuming that all of the 'lapse functions' are positive ²:

$$\begin{aligned} s_a s^a &= 1 = \left(\frac{B}{\bar{B}} \bar{s}_a - \psi n_a \right) \left(\frac{B}{\bar{B}} \bar{s}^a - \psi n^a \right) , \\ 1 &= \left(\frac{B}{\bar{B}} \right)^2 - \psi^2 , \\ \frac{\bar{B}}{B} &= \frac{1}{\sqrt{1 + \psi^2}} . \end{aligned} \tag{D.0.5}$$

Continuing this process,

$$\begin{aligned} \bar{n}_a \bar{n}^a &= \left(\frac{\bar{\alpha}}{\alpha} (1 + \psi^2) n_a - \left(\frac{\bar{\alpha} B}{\alpha \bar{B}} \right) \psi \bar{s}_a \right) \left(\frac{\bar{\alpha}}{\alpha} (1 + \psi^2) n^a - \left(\frac{\bar{\alpha} B}{\alpha \bar{B}} \right) \psi \bar{s}^a \right) , \\ -1 &= - \left(\frac{\bar{\alpha}}{\alpha} \right)^2 (1 + \psi^2)^2 + \left(\frac{\bar{\alpha} B}{\alpha \bar{B}} \right)^2 \psi^2 , \\ &= - \left(\frac{\bar{\alpha}}{\alpha} \right)^2 [1 + 2\psi^2 + \psi^4 - (1 + \psi^2) \psi^2] , \\ \frac{\bar{\alpha}}{\alpha} &= \frac{1}{\sqrt{1 + \psi^2}} = \frac{\bar{B}}{B} . \end{aligned} \tag{D.0.6}$$

While the condition $\bar{n}_a s^a = 0$ gives no new information.

Now, substituting equation (D.0.6) back into equations (D.0.3) and (D.0.4) yields

²And if they are not, minus signs can be absorbed into the R and τ in order to enforce this positivity

$$\begin{aligned}
\bar{n}_a &= \sqrt{1 + \psi^2} n_a - \psi \bar{s}_a, \\
s_a &= \sqrt{1 + \psi^2} \bar{s}_a - \psi n_a.
\end{aligned} \tag{D.0.7}$$

Which is clearly a Lorentz boost with boost parameter ϕ given by $\psi = \sinh(\phi)$. Therefore, inverting the order in which one splits in a 1+1+2 split formalism will be equivalent to a boost when done the other way.

And, as a consistency check, we compute:

$$\begin{aligned}
\bar{n}^a \bar{n}^b - s^a s^b &= \left(\sqrt{1 + \psi^2} n^a - \psi \bar{s}^a \right) \left(\sqrt{1 + \psi^2} n^b - \psi \bar{s}^b \right), \\
&\quad - \left(\sqrt{1 + \psi^2} \bar{s}^a - \psi n^a \right) \left(\sqrt{1 + \psi^2} \bar{s}^b - \psi n^b \right), \\
&= (1 + \psi^2) n^a n^b - \psi \sqrt{1 + \psi^2} (n^a \bar{s}^b + n^b \bar{s}^a) + \psi^2 s^a s^b \\
&\quad - (1 + \psi^2) \bar{s}^a \bar{s}^b + \psi \sqrt{1 + \psi^2} (\bar{s}^a n^b + \bar{s}^b n^a) - \psi^2 n^a n^b, \\
&= n^a n^b - \bar{s}^a \bar{s}^b.
\end{aligned} \tag{D.0.8}$$

So, as promised, our two projection operators are identical.

D.1 Example: calculating the velocity parameter of Kerr Spacetime

Now, for the sake of completeness, let us calculate the value of the boost parameter ψ for the Kerr spacetime. Take the metric to be given by:

$$g_{ab} = \begin{pmatrix} -\left(1 - \frac{2Mr}{B}\right) & \frac{2Mr}{B} & \frac{2Mr}{B}a \sin^2(\theta) & 0 \\ \frac{2Mr}{B} & 1 + \frac{2Mr}{B} & -\left(1 + \frac{2Mr}{B}\right)a \sin^2(\theta) & 0 \\ \frac{2Mr}{B}a \sin^2(\theta) & -\left(1 + \frac{2Mr}{B}\right)a \sin^2(\theta) & \left(A + \frac{2Mr a^2 \sin^2(\theta)}{B}\right) \sin^2(\theta) & 0 \\ 0 & 0 & 0 & B \end{pmatrix}. \quad (\text{D.1.1})$$

$$g^{ab} = \begin{pmatrix} -\left(1 + \frac{2Mr}{B}\right) & \frac{2Mr}{B} & 0 & 0 \\ \frac{2Mr}{B} & \frac{A-2Mr}{B} & \frac{a}{B} & 0 \\ 0 & \frac{a}{B} & \frac{1}{B \sin^2(\theta)} & 0 \\ 0 & 0 & 0 & B \end{pmatrix}. \quad (\text{D.1.2})$$

In coordinates labeled by (t, r, ϕ, θ) , and all other parameters labeled as in Appendix A. Take the function $\tau = -t$, and the function $R = r$. Then, we obtain via a simple reading from the above expression of the inverse to the metric tensor:

$$\begin{aligned}
n_a &= \left(-\sqrt{\frac{B}{B+2Mr}}, 0, 0, 0 \right) & s_a &= \left(0, \sqrt{\frac{B}{A-2Mr}}, 0, 0 \right), \\
\psi = n^a s_a &= g^{ab} n_a s_b = -\frac{2Mr}{B} \left(\sqrt{\frac{B}{B+2Mr}} \right) \sqrt{\frac{B}{A-2Mr}}, \\
&= -\frac{2Mr}{\sqrt{(B+2Mr)(A-2Mr)}} = \sinh(\phi).
\end{aligned} \tag{D.1.3}$$

Then, solving $1 = \cosh^2(\phi) - \sinh^2(\phi)$ gives:

$$\cosh(\phi) = \sqrt{\frac{AB + 2Mr a^2 \sin^2(\theta)}{(A-2Mr)(B+2Mr)}}. \tag{D.1.4}$$

Which immediately allows us, in analogy with Special Relativity, to find the velocity parameter v ,

$$v = \tanh(\phi) = -\frac{\sinh(\phi)}{\cosh(\phi)} = \frac{2Mr}{\sqrt{AB + 2Mr a^2 \sin^2(\theta)}}. \tag{D.1.5}$$

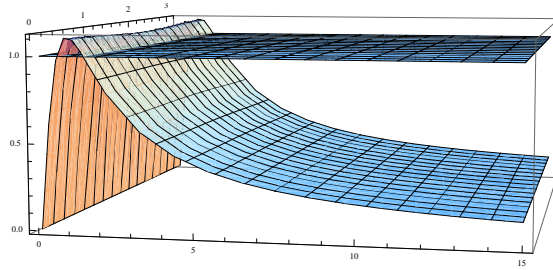


Figure D.1: The velocity parameter v plotted against r/M and θ . The plane is the surface $f = 1$ and $a = 0.9$.

This term can be interpreted as the velocity of the 3-surface of constant

R relative to the 3-surface of constant τ . Note that it is zero on the surface at spacelike infinity, as $r \rightarrow \infty$. Since $g^{rt} = \frac{\beta^r}{\alpha^2} \rightarrow 0$ for asymptotically flat spacetimes, this result is only sensitive to asymptotic flatness, and not the details of the Kerr solution. Furthermore, for the case of asymptotically flat spacetimes, one can use the fact that $v = 0$ at infinity in order to interpret v as the velocity of a point's reference frame with respect to conformal spacelike infinity. In order to bolster this interpretation, note that v takes the value of 1 for all points on either horizon $r = M \pm \sqrt{M^2 - a^2}$, is greater than one in the region between the two horizons, and less than 1 everywhere else³. A *Mathematica* plot of v for $a = .9 M$ in units of r/M is included in figure (D.1).

³As another aside, note that v takes the value 0 for $r = 0$. Observers inside the inner horizon cannot access the region outside the horizon, making it somewhat nonsensical to talk about “local speeds of spacetime” relative to conformal infinity. One could, however, interpret v as the velocity of the reference frames relative to the point at the center of the Kerr ring.

Bibliography

- [1] T. Regge and C. Teitelboim, “Role of surface integrals in the hamiltonian formulation of general relativity,” *Annals of Physics* (Jan, 1974) . <http://adsabs.harvard.edu/abs/1974AnPhy..88..286R>.
- [2] G. B. Cook, “Corotating and irrotational binary black holes in quasi-circular orbits,” *arXiv gr-qc* (Jan, 2001) , [gr-qc/0108076v2](http://arxiv.org/abs/gr-qc/0108076v2).
<http://arxiv.org/abs/gr-qc/0108076v2>.
- [3] E. Schnetter, B. Krishnan, and F. Beyer, “Introduction to dynamical horizons in numerical relativity,” *arXiv gr-qc* (Apr, 2006) , [gr-qc/0604015v2](http://arxiv.org/abs/gr-qc/0604015v2). <http://arxiv.org/abs/gr-qc/0604015v2>.
- [4] J. L. Jaramillo, J. A. V. Kroon, and E. Gourgoulhon, “From geometry to numerics: interdisciplinary aspects in mathematical and numerical relativity,” *arXiv gr-qc* (Dec, 2007) , [0712.2332v2](http://arxiv.org/abs/0712.2332v2).
<http://arxiv.org/abs/0712.2332v2>.
- [5] J. L. Jaramillo, M. Ansorg, and F. Limousin, “Numerical implementation

- of isolated horizon boundary conditions,” *arXiv gr-qc* (Oct, 2006) ,
[gr-qc/0610006v2](http://arxiv.org/abs/gr-qc/0610006v2). <http://arxiv.org/abs/gr-qc/0610006v2>.
- [6] J. L. Jaramillo, E.ourgoulhon, I. Cordero-Carrion, and J. M. Ibanez,
 “Trapping horizons as inner boundary conditions for black hole
 spacetimes,” *arXiv gr-qc* (Sep, 2007) , [0709.2842v1](http://arxiv.org/abs/0709.2842v1).
<http://arxiv.org/abs/0709.2842v1>.
- [7] A. Ashtekar, J. Baez, and K. Krasnov, “Quantum geometry of isolated
 horizons and black hole entropy,” *arXiv gr-qc* (May, 2000) ,
[gr-qc/0005126v1](http://arxiv.org/abs/gr-qc/0005126v1). <http://arxiv.org/abs/gr-qc/0005126v1>.
- [8] A. Ashtekar, C. Beetle, O. Dreyer, S. Fairhurst, B. Krishnan,
 J. Lewandowski, and J. Wisniewski, “Generic isolated horizons and their
 applications,” *arXiv gr-qc* (Jun, 2000) , [gr-qc/0006006v2](http://arxiv.org/abs/gr-qc/0006006v2).
<http://arxiv.org/abs/gr-qc/0006006v2>.
- [9] V. Guillemin and A. Pollack, *Differential Topology*. Prentice-Hall, 1974.
- [10] C. Misner, K. Thorne, and J. A. Wheeler, *Gravitation*. W.H. Fredman
 and Company, 1971.
- [11] R. Arnowitt, S. Deser, and C. W. Misner, “The dynamics of general
 relativity,” *arXiv gr-qc* (Jan, 2004) , [gr-qc/0405109v1](http://arxiv.org/abs/gr-qc/0405109v1).
<http://arxiv.org/abs/gr-qc/0405109v1>.
- [12] S. W. Hawking and G. F. R. Ellis, *The Large-Scale Structure of
 Space-Time*. Cambridge University Press, 1973.

- [13] R. Wald, *General Relativity*. University of Chicago Press, 1984.
- [14] S. W. Hawking.
- [15] E. Poisson, *A Relativists Toolkit*. Cambridge University Press, 2004.
- [16] A. Ashtekar and B. Krishnan, “Isolated and dynamical horizons and their applications,” *arXiv gr-qc* (Jul, 2004) , [gr-qc/0407042v3](https://arxiv.org/abs/gr-qc/0407042v3).
<http://arxiv.org/abs/gr-qc/0407042v3>.
- [17] S. A. Hayward, “Angular momentum conservation for dynamical black holes,” *arXiv gr-qc* (Jan, 2006) , [gr-qc/0609008v2](https://arxiv.org/abs/gr-qc/0609008v2).
<http://arxiv.org/abs/gr-qc/0609008v2>.
- [18] P. C. Vaidya, “Nonstatic solutions of einstein’s field equations for spheres of fluids radiating energy,” *Physical Review* **83** (1951) no. 2, 10–17.
- [19] A. B. Nielsen, M. Jasiulek, B. Krishnan, and E. Schnetter, “The slicing dependence of non-spherically symmetric quasi-local horizons in vaidya spacetimes,” *arXiv gr-qc* (Jul, 2010) , [1007.2990v1](https://arxiv.org/abs/1007.2990v1).
<http://arxiv.org/abs/1007.2990v1>. 10 pages, 5 figures.
- [20] L. Landau and E. Lifshitz, *Mechanics*. Reed Educational and Professional Publishing, third ed., 1981.
- [21] J. D. Brown, S. R. Lau, and J. W. York, “Action and energy of the gravitational field,” *arXiv gr-qc* (Jan, 2002) , [gr-qc/0010024v3](https://arxiv.org/abs/gr-qc/0010024v3).
<http://arxiv.org/abs/gr-qc/0010024v3>.

- [22] B. O’Neil, *Semi-Riemannian Geometry: with applications to relativity*. Academic Press, 1983.
- [23] S. A. Hayward, “Energy and entropy conservation for dynamical black holes,” *arXiv gr-qc* (Aug, 2004) , [gr-qc/0408008v2](https://arxiv.org/abs/gr-qc/0408008v2).
<http://arxiv.org/abs/gr-qc/0408008v2>.
- [24] V. P. Frolov, “Embedding of the kerr-newman black hole surface in euclidean space,” *arXiv gr-qc* (Jan, 2006) , [gr-qc/0601104v1](https://arxiv.org/abs/gr-qc/0601104v1).
<http://arxiv.org/abs/gr-qc/0601104v1>.
- [25] L. Fatibene, M. Francaviglia, and S. Mercadante, “Noether symmetries and covariant conservation laws in classical, relativistic and quantum physics,”.
- [26] L. Szabados, “" quasi-local energy-momentum and angular momentum in gr: A review article,” *Living Rev. Relativity* (Jan, 2004) .
<http://www.emis.ams.org/journals/LRG/Articles/lrr-2004-4/download/lrr-2004-4BW.pdf>.
- [27] E. Gourgoulhon and J. L. Jaramillo, “Trapping horizons as inner boundary conditions for black hole spacetimes,”.
- [28] L. B. Szabados, “Quasi-local energy-momentum and angular momentum in gr: A review article,” *Living Reviews in Relativity* **7** (Jun, 2004) 1–140.
- [29] S. A. Hayward, “Angular momentum conservation for uniformly

- expanding flows,” *arXiv gr-qc* (Jan, 2006) , [gr-qc/0611027v1](http://arxiv.org/abs/gr-qc/0611027v1).
<http://arxiv.org/abs/gr-qc/0611027v1>.
- [30] G. Allemandi, M. Francaviglia, and M. Raiteri, “The first law of isolated horizons via noether theorem,” *arXiv gr-qc* (Oct, 2001) ,
[gr-qc/0110104v1](http://arxiv.org/abs/gr-qc/0110104v1). <http://arxiv.org/abs/gr-qc/0110104v1>.
- [31] A. Ashtekar and R. Hansen, “A unified treatment of null and spatial infinity in general relativity. i. universal . . .,” *Journal of Mathematical Physics* (Jan, 1978) . <http://link.aip.org/link/?JMAPAQ/19/1542/1>.
- [32] E. Comay, “Electromagnetic energy-momentum tensor and elementary classical point charges,” *International Journal of Theoretical Physics* **30** (Oct, 1990) 1461–1472.
- [33] A. Ashtekar and B. Krishnan, “Dynamical horizons and their properties,” *arXiv gr-qc* (Jan, 2003) , [gr-qc/0308033v4](http://arxiv.org/abs/gr-qc/0308033v4).
<http://arxiv.org/abs/gr-qc/0308033v4>.
- [34] J. Charap and J. Nelson, “Surface integrals and the gravitational action,” *Journal of Physics A: Mathematical and General* (Jan, 1983) .
<http://www.iop.org/EJ/abstract/0305-4470/16/8/013>.
- [35] A. Ashtekar, S. Fairhurst, and B. Krishnan, “Isolated horizons: Hamiltonian evolution and the first law,” *arXiv gr-qc* (Jan, 2000) ,
[gr-qc/0005083v3](http://arxiv.org/abs/gr-qc/0005083v3). <http://arxiv.org/abs/gr-qc/0005083v3>.

- [36] E. Schnetter and B. Krishnan, “Non-symmetric trapped surfaces in the schwarzschild and vaidya spacetimes,” *arXiv gr-qc* (Nov, 2005) , [gr-qc/0511017v3](http://arxiv.org/abs/gr-qc/0511017v3). <http://arxiv.org/abs/gr-qc/0511017v3>.
- [37] L. Brewin, “A simple expression for the adm mass,” *arXiv gr-qc* (Jan, 2006) , [gr-qc/0609079v1](http://arxiv.org/abs/gr-qc/0609079v1). <http://arxiv.org/abs/gr-qc/0609079v1>.
- [38] R. Wald, “Lagrangians and hamiltonians in classical field theory,”.
- [39] S. Slobodov, “Line profiles of accretion disks around black holes in schwarzschild-de sitter and einstein-yang-mills spacetimes,”.
- [40] M. Francaviglia and M. Raiteri, “Hamiltonian, energy and entropy in general relativity with non-orthogonal boundaries,” *arXiv gr-qc* (Jul, 2001) , [gr-qc/0107074v1](http://arxiv.org/abs/gr-qc/0107074v1). <http://arxiv.org/abs/gr-qc/0107074v1>.
- [41] S. A. Hayward, “Angular momentum and conservation laws for dynamical black holes,” *arXiv gr-qc* (Jan, 2006) , [gr-qc/0601058v1](http://arxiv.org/abs/gr-qc/0601058v1). <http://arxiv.org/abs/gr-qc/0601058v1>.
- [42] D. M. Eardley, “Black hole boundary conditions and coordinate conditions,” *arXiv gr-qc* (Mar, 1997) , [gr-qc/9703027v2](http://arxiv.org/abs/gr-qc/9703027v2). <http://arxiv.org/abs/gr-qc/9703027v2>.
- [43] A. Corichi, U. Nucamendi, and D. Sudarsky, “Einstein-yang-mills isolated horizons: Phase space, mechanics, hair and conjectures,” *arXiv gr-qc* (Feb, 2000) , [gr-qc/0002078v1](http://arxiv.org/abs/gr-qc/0002078v1). <http://arxiv.org/abs/gr-qc/0002078v1>.

- [44] A. Ashtekar and A. Magnon-Ashtekar, “Energy-momentum in general relativity,” *Physical Review Letters* (Jan, 1979) .
<http://link.aps.org/doi/10.1103/PhysRevLett.43.181>.
- [45] A. Ashtekar, C. Beetle, and J. Lewandowski, “Mechanics of rotating isolated horizons,” *arXiv gr-qc* (Mar, 2001) , [gr-qc/0103026v2](http://arxiv.org/abs/gr-qc/0103026v2).
<http://arxiv.org/abs/gr-qc/0103026v2>.
- [46] C.-H. Wang, “Gravitational radiations of generic isolated horizons and non-rotating dynamical horizons from asymptotic expansions,” *arXiv gr-qc* (Jun, 2009) , [0906.1551v1](http://arxiv.org/abs/0906.1551v1).
<http://arxiv.org/abs/0906.1551v1>.
- [47] J. Stryla, “Isometric embeddings of the reduced horizon of a kerr-newman black hole into euclidean 3-space,” *Classical and Quantum Gravity* **15** (Jul, 1998) 1–9.
- [48] A. Ashtekar, C. Beetle, and S. Fairhurst, “Mechanics of isolated horizons,” *arXiv gr-qc* (Jul, 1999) , [gr-qc/9907068v2](http://arxiv.org/abs/gr-qc/9907068v2).
<http://arxiv.org/abs/gr-qc/9907068v2>.
- [49] E. Poisson and I. Vlasov, “Geometry and dynamics of a tidally deformed black hole,” *arXiv gr-qc* (Oct, 2009) , [0910.4311v1](http://arxiv.org/abs/0910.4311v1).
<http://arxiv.org/abs/0910.4311v1>.
- [50] E. Gourgoulhon and J. L. Jaramillo, “A 3+1 perspective on null

hypersurfaces and isolated horizons," *arXiv* **gr-qc** (Mar, 2005) ,
gr-qc/0503113v2. <http://arxiv.org/abs/gr-qc/0503113v2>.

Vita

Jerry Schirmer was born in St. Charles, Missouri to Anne and Michael Schirmer. He attended Marquette High School in Chesterfield Missouri, eventually attending college at Truman State University, where he graduated Magna Cum Laude and Phi Beta Kappa with degrees in Physics, Philosophy & Religion, and Political Science. He has a deep interest in the role that science and the scientist has in society.

Permanent Address: 2736 Stonewall Sta.

St. Charles, MO 63303

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